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LETTER TO THE EDITOR

Quantum algebraic structures compatible with the harmonic oscillator Newton equation

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Abstract. We study some of the algebraic structures that are compatible with the quantization of the harmonic oscillator through its Newton equation. Examples of such structures are given; they include undeformed and q-deformed oscillators, as well as the $SU(2)$ and the deformed $SU_q(2)$ Lie algebras, which appear in a variety of physical models.

The question of whether the equations of motion of a system determine the quantum mechanical commutation relations was answered in the negative by Wigner some 50 years ago [1]. Taking the well-behaved example of the harmonic oscillator (with unit mass and restitution constant ω), with its classical Hamiltonian $h = \frac{1}{2}(p^2 + \omega^2 q^2)$ and momentum $p = \dot{q}$, and requiring that the energy spectrum of the associated quantum operator H is bounded from below, Wigner showed that the commutator between the momentum and position operators is satisfied not only by the canonical Heisenberg commutation relations $[P, Q] = -\mathrm{i}$ (in units where $\hbar = 1$), but by (at least one) more general solution written as $([P, Q] + \mathrm{i})^2 = -(2E_0 - 1)^2$, where E_0 is a real constant characteristic of the solution. Informed of other solutions that have been useful in the recent mathematical, physical and optical literature (such as q-deformed systems and phase-space analysis of discrete data sets [2]), we present here other general algebraic structures which are compatible with the quantization of the harmonic oscillator, whose discrete spectrum need not be half-infinite and equally spaced, and whose foundation is the Newton equation of motion.

The time evolution of a classical system with potential $V(q)$ and unit mass is determined by the Newton equation $\ddot{q} = -\partial V(q)/\partial q$, which is a differential equation of second order in the particle position coordinate $q \in \mathbb{R}$. Its solutions $q(t)$ depend on the initial position and momentum of the particle. Quantization into a one-parameter evolution Lie group of operators, is achieved through replacing the time derivatives of the classical quantities by commutators with its quantum Hamiltonian operator, times $-\mathrm{i}$. For the harmonic oscillator of frequency ω [3], whose Newton equation is $\ddot{q} = -\omega^2 q$, the resulting quantum equation is thus

$$[H, [H, Q]] = \omega^2 Q \tag{1}$$

where the inner commutator is by definition the momentum operator, and the outer commutator specifies the system, i.e.,

$$P = \mathrm{i}[H, Q] \qquad [H, P] = \mathrm{i}\omega^2 Q. \tag{2}$$

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The last two expressions are the Hamilton equations of motion in Lie evolution form. They separate Newton's equation (1) into one geometric and one dynamic factor.

Observe that the commutator $[Q, P]$ is thus far unspecified. If we require that the three operators, Q, P and H—and also the unity operator—close into an associative algebra, then they must satisfy the Jacobi identity,

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0. \tag{3}$$

In view of equations (2), the Jacobi identity for these three operators requires that

$$[H, [Q, P]] = 0. \tag{4}$$

From here it is evident that the heretofore unspecified commutator can be any function f of H,

$$[Q, P] = \mathrm{i} f(H). \tag{5}$$

In particular, when $f(H) = 1$, one recovers the Heisenberg–Weyl algebra and the common quantum mechanical treatment of the harmonic oscillator.

As usual, for convenience we choose units where $\omega = 1$ and introduce the complex linear combinations

$$A = \frac{1}{\sqrt{2}}(Q + \mathrm{i}P) \qquad A^{+} = \frac{1}{\sqrt{2}}(Q - \mathrm{i}P). \tag{6}$$

With the aid of equations (2) only, one finds the well known commutators

$$[H, A] = -A \qquad [H, A^{+}] = A^{+}. \tag{7}$$

From here it follows immediately that their product commutes with the Hamiltonian:

$$[A^{+}A, H] = 0. \tag{8}$$

Again we remark that the solution to this can be

$$A^{+}A = g(H) \tag{9}$$

with a second function g. If this function g is analytic, then

$$AA^{+}A = A\, g(H) = g(H + 1)A. \tag{10}$$

The commutator between the two operators (6) can then be found to be

$$[A, A^{+}] = AA^{+} - A^{+}A = g(H + 1) - g(H). \tag{11}$$

Strictly speaking this equation holds only on the range of A. We will assume that it holds in general.

On the other hand, from the Jacobi-derived condition (5), we also find their commutator, but in the form

$$[A, A^{+}] = \tfrac{1}{2}[Q + \mathrm{i}P, Q - \mathrm{i}P] = -\mathrm{i}[Q, P] = f(H). \tag{12}$$

From the two previous equations we obtain the important finite-difference relation

$$f(H) = g(H + 1) - g(H). \tag{13}$$

In the case $f(H) = 1$ of the Heisenberg–Weyl Lie algebra, (13) implies that $g(H) = H + \text{constant}$, and this in turn bestows upon A^{+} and A the role of creation and annihilation operators, which serve to build an irreducible representation of the algebra on a basis $|n\rangle \sim (A^{+})^{n}|0\rangle$, starting from an assumed ground state $|0\rangle$. This is the eigenbasis of the *number* operator $N = g(H) = H - \frac{1}{2}$, whose spectrum is $\{0, 1, 2, \ldots\}$.

In the general case (9), when f and g satisfy only (13), we assume that there exists a ground state $|0\rangle$ such that

$$A|0\rangle = 0 \qquad H|0\rangle = E_0|0\rangle \tag{14}$$

and introduce (as usual) the *number* operator

$$N = H - E_0. \tag{15}$$

From f and g in equations (5) and (9), it is convenient to define the deformed operators

$$\tilde{f}(N) = f(H) = f(N + E_0) = AA^+ - A^+A \tag{16}$$

$$\tilde{g}(N) = g(H) = g(N + E_0) = A^+A \tag{17}$$

which satisfy the same difference relation (13) as their untilded partners. We do expect that these functions contain some parameter or parameters, with a limit that leads to the 'undeformed' Heisenberg–Weyl quantum harmonic oscillator as a limit case. These structures have been named *generalized oscillators* [4]. For 'oscillator-like' models it is natural to regard $\tilde{g}(N)$ in (17) as a (new) *deformed number operator*, because of the common form of its right-hand side. For '$SU(2)$-like' models on the other hand [3, 5], where the Hamiltonian is an element of the Lie algebra, it is the choice of $\tilde{f}(0) - \tilde{f}(N)$ in (16) which naturally deserves the name of *deformed number operator*, because it numbers the eigenstates of the Hamiltonian starting from the ground one with eigenvalue zero.

We now present several examples of algebraic structures which satisfy equation (13), and are therefore compatible with the one-dimensional Newton equation (1).

(1) The undeformed structure with

$$\tilde{f}(N) = 1 \qquad \tilde{g}(N) = N \tag{18}$$

leads to the well known Heisenberg–Weyl algebra of the traditional quantum harmonic oscillator, one of whose salient characteristics is the lower-bound discrete energy spectrum of the Hamiltonian $H|n\rangle = E_n|n\rangle$, which is half-infinite and equally spaced,

$$E_n = n + \tfrac{1}{2} \qquad n \in \{0, 1, 2, \ldots\}. \tag{19}$$

(2) The $SU(2)$ case [3, 5] contains one discrete parameter, $j \in \{0, \frac{1}{2}, 1, \frac{3}{2}, \ldots\}$,

$$\tilde{f}(N) = 2j - 2N \qquad \tilde{g}(N) = N(2j + 1 - N). \tag{20}$$

The index j labels the $SU(2)$ irreducible representation; j has the obvious interpretation of spin, but in applications to discrete optics [5], this index characterizes finite signals of $2j + 1$ data points. The eigenvalues of the number operator are

$$\tfrac{1}{2}[\tilde{f}(0) - \tilde{f}(n)] \qquad n \in \{0, 1, \ldots, 2j\}. \tag{21}$$

The energy spectrum of the finite $SU(2)$ oscillator is similar to (19) in that it is bounded from below and equally spaced; however, it has an *upper* bound too.

(3) The q-oscillator [6] contains one continuous real parameter $q \in \mathbb{R}$,

$$\tilde{f}(N) = q^N \qquad \tilde{g}(N) = \frac{1 - q^N}{1 - q} \tag{22}$$

and the spectrum of the deformed number operator is

$$\tilde{g}(n) = \frac{1 - q^n}{1 - q} \qquad n \in \{0, 1, 2, \ldots\} \qquad q \in \mathbb{R}. \tag{23}$$

(4) The q-oscillator of Biedenharn and Macfarlane [7] (see also [8, 9]), also for one parameter $q \in \mathbb{R}$, is given by

$$\tilde{f}(N) = \frac{q^{N+1/2} + q^{-N-1/2}}{q^{1/2} + q^{-1/2}} \qquad \tilde{g}(N) = [N]_q = \frac{q^N - q^{-N}}{q - q^{-1}}. \tag{24}$$

The spectrum of N is the same as in the previous case, but the spectrum of the deformed number operator is here $[n]_q$.

(5) The $SU_q(2)$ q-algebra [10], with one discrete and one continuous parameter, j and q as above,

$$\tilde{f}(N) = [2(j-N)]_q = \frac{q^{2(j-N)} - q^{2(N-j)}}{q - q^{-1}} \qquad \tilde{g}(N) = [N]_q\,[2j+1-N]_q. \tag{25}$$

If the Hamiltonian is a shifted deformed number operator, its spectrum will be $\tilde{f}(0) - \tilde{f}(N)$, but with a finite number of levels, $n \in \{0, 1, \ldots, 2j\}$, $q \in \mathbb{R}$.

The above five examples of structures, and indeed any other algebraic structures obeying (16), (17), are compatible with the one-dimensional Newton equation (1). However, many physical applications require multidimensional harmonic oscillators, so it is useful to investigate the additional constraints on the algebra brought by this generalization. We first consider quantum algebraic structures which are compatible with the two-dimensional Newton equation. For coordinates 1 and 2, denote the associated undeformed and deformed number operators by N_1, N_2 and $\tilde{g}(N_1)$, $\tilde{g}(N_2)$. Each deformed number operator of this system and the total deformed number operator should have the same eigenvalues as the deformed number operator of a one-dimensional system. Therefore, for $N = N_1 + N_2$, it is most natural to write:

$$\tilde{g}(N) = \tilde{g}(N_1) + \tilde{g}(N_2) = A_1^+ A_1 + A_2^+ A_2 \tag{26}$$

and this implies that $\tilde{g}(N) = N$, as in the Heisenberg–Weyl case. However, it is possible to modify this equation so that the deformed structure also be kept for the multidimensional case. We will specifically consider

$$G(N_1, N_2) = \phi(N_2) A_1^+ A_1 + \phi(N_1) A_2^+ A_2 \tag{27}$$

with $\phi(0) = 1$ such that for the ground state in the second coordinate, i.e., for $N_2 = 0 = A_2^+ A_2$, this reduces to (17). Note that the two oscillators are treated in a symmetric way.

We define new deformed annihilation operators along the two coordinates,

$$C_1 = \sqrt{\phi(N_2)}\,A_1 \qquad C_2 = \sqrt{\phi(N_1)}\,A_2 \tag{28}$$

and require that they commute: $[C_1, C_2] = 0$. This definition is analogous to (but distinct from) the definition in [11] used for the construction of a quantum-group-invariant q-oscillator from commuting one-dimensional q-oscillators. Upon using the equation $A_1 f(N_1) = f(N_1+1) A_1$, which is valid for any analytic function f (cf the one-dimensional case (10)), we can write

$$G(N_1, N_2) = C_1^+ C_1 + C_2^+ C_2. \tag{29}$$

From $C_1 C_2 = C_2 C_1$ we obtain a difference equation which, together with $\phi(0) = 1$, leads to

$$\frac{\phi(N_1+1)}{\phi(N_1)} = \frac{\phi(N_2+1)}{\phi(N_2)} = q \Rightarrow \phi(N) = q^N. \tag{30}$$

Now we find the function $\tilde{g}$ which determines the structure

$$\tilde{g}(N_1 + N_2) = q^{N_2} \tilde{g}(N_1) + q^{N_1} \tilde{g}(N_2). \tag{31}$$

For $N_2 = 1$, $\tilde{g}(N_1+1) = q\tilde{g}(N_1) + q^{N_1}\tilde{g}(1)$ results in the difference equation $\tilde{g}(N_1+1)/\tilde{g}(1) = q\tilde{g}(N_1)/\tilde{g}(1) + q^{N_1}$, whose solutions are

$$\tilde{g}(N) = \tilde{g}(1)(Nq^{N-1} + \kappa q^N) \tag{32}$$

with κ a constant. But, since (14)–(17) require that $\tilde{g}(0) = 0$, it follows that $\kappa = 0$; finally, we can set $g(1) = 1$ by rescaling the operators, and display the solution as

$$\tilde{g}(N) = N\,q^{N-1}. \tag{33}$$

The original two-dimensional deformed number operator can therefore be written as

$$G(N_1, N_2) = C_1^+ C_1 + C_2^+ C_2 = N q^{N-1} \qquad N = N_1 + N_2. \tag{34}$$

In the three-dimensional case, when there are three coordinates labelled $k = 1, 2, 3$, and corresponding operators A_k, N_k and C_k, which commute for distinct subindices, the above construction generalizes to

$$C_1 = q^{(N_2+N_3)/2} A_1 \qquad C_2 = q^{(N_3+N_1)/2} A_2 \qquad C_3 = q^{(N_1+N_2)/2} A_3. \tag{35}$$

For generic dimension d, the algebraic structure of commutation relations

$$C_j C_k = C_k C_j \tag{36}$$

$$C_j C_k^+ - q C_k^+ C_j = q^N \delta_{j,k} \tag{37}$$

leads to the total deformed number operator given by

$$G(N_1, \ldots, N_d) = C_1^+ C_1 + \cdots + C_d^+ C_d = \tilde{g}(N) = N q^{N-1}. \tag{38}$$

for $N = N_1 + \cdots + N_d$. The commutation relations (36) and the total deformed number operators (37) enjoy the unitary $U(d)$ symmetry of the ordinary quantum harmonic oscillator, which acts on the (covariant) creation and annihilation operators C_k^+, C_k, and survives deformation.

The multidimensional oscillator (36), (37) is interesting in that it is related to several of the algebraic structures considered above. When q^N is absent from the right-hand side of the commutation relation (37), that equation defines the Coon–Yu–Baker oscillator [12] used in the factorization of the q-deformed dual resonance model amplitudes. However, as shown in [13], in this case equation (36) is incompatible with (37). Finally, the two-parameter oscillator [14] in the limit when the two parameters q_1 and q_2 approach each other (i.e., $q_1 = q_2 = q$) also gives (37), and this limit reproduces the model with the parameter $p = 1/q$, discussed earlier in [15]. This same deformed oscillator arises when constructing a new family of boson coherent states in [16], which uses a special q-extension of the exponential function e^z. As the other references indicate, there are models of various algebraic structures also in quantum optics, electromagnetic field quantization, and signal propagation in waveguides. There are various assignments between operators and physical observables, which we do not attempt to detail and tabulate here. The five example cases of one-dimensional systems and their ensuing generalization to higher dimensions are the results we wish to present here on quantum structures which are compatible with Newton's equation.

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