Structure of the set of paraxial optical systems

R. Simon

The Institute of Mathematical Sciences, C.I.T. Campus, Tharamani, Chennai 600 113, India

Kurt Bernardo Wolf

Centro de Ciencias Físicas, Universidad Nacional Autónoma de México, Apartado Postal 48-3, 62251 Cuernavaca, Mexico

Received May 26, 1999; revised manuscript received October 1, 1999; accepted October 5, 1999

The set of paraxial optical systems is the manifold of the group of symplectic matrices. The structure of this group is nontrivial: It is not simply connected and is not of an exponential type. Our analysis clarifies the origin of the metaplectic phase and the inherent limitations for optical map fractionalization. We describe, for the first time to our knowledge, an image girator and a cross girator whose geometric and wave implementations are of interest. © 2000 Optical Society of America [S0740-3232(00)00102-2]

OCIS codes: 080.2730, 000.3870.

1. INTRODUCTION: PARAXIAL OPTICAL SYSTEMS

Paraxial optics is a well-defined mathematical structure in which geometric and wave optics are in exact, though not trivial, correspondence. Roughly said, optical elements are represented by symplectic matrices that multiply in a definite order as light traverses them. In geometric optics, these matrices act on a column vector that contains the canonical coordinates of phase space and whose points label straight lines in space—the light rays of geometric optics. In wave optics, on the other hand, the elements act through integral transforms on a Hilbert space of functions that describe a scalar wave field. The latter are actually in two-to-one correspondence with the former, in a way that is often misunderstood in the current optical literature. Questions about the realizability of a given transformation and its possible fractionalization are often plagued with *ad hoc* solutions and errors that could be easily avoided by a careful reading of the mathematical literature. Doing so is not an easy proposition, though, because *abcd*-matrix theory has entered mainstream optics only since the beginning of the 1980's, and the previous decades' research has to be translated into the current language of optics.

The mathematical structure transcends the geometrical model analyzed here, moreover. A thorough understanding of the symplectic groups is based well on this geometric-optical model, which is their simplest realization by matrices. Other realizations include paraxial wave (or Fourier–Gauss) optics, as mentioned above, in which the same group is represented by operators (which act on wave fields). Optical information processing is served when the waveguides and image-producing devices are reinterpreted as Fourier transformers and time lenses. Further, the linear theory of quantum optics reinterprets the effect of magnifiers by squeezing, and fractional Fourier transformations by time evolution of the field.

In this paper we address specifically two- and three-dimensional paraxial optical systems characterized by two- and four-dimensional symplectic matrices and give enough elements to permit the theory to be generalized to any dimension. In Section 2 we show that the preservation of a Hamiltonian system under linear transformations implies the symplectic conditions that characterize the Lie group $\mathrm{Sp}(2N, \mathfrak{R})$ of matrices. We find the Iwasawa decomposition[1] particularly suited for optical applications because the subgroups of the symplectic group can be identified with basic constituents and arrangements: lenses, magnifiers, and various phase-space rotators. In this respect, the Iwasawa decomposition is preferable to the Bargmann decomposition[2] that is common in the mathematical literature.[3] The two-dimensional case $\mathrm{Sp}(2, \mathfrak{R})$ is analyzed in Section 3, and in Section 4 its elements are realized as optical arrangements made from positive displacements (free propagation through a homogeneous medium) and lenses built into magnifiers and phase-space rotators; the latter bear the metaplectic winding number. The fractionalization of the Fourier transform has been the subject of much interest, yet in Section 5 we determine the class of paraxial systems that cannot be fractionalized.

Three-dimensional paraxial optics is addressed in Section 6 with the study of the symplectic group of 4×4 matrices $\mathrm{Sp}(4, \mathfrak{R})$. In the Iwasawa decomposition we separate the optical constituents into astigmatic lenses, pure magnifiers, and a subgroup of unitary rotators and gyrators of phase space. The axisymmetrical fractional Fourier transformation carries the metaplectic winding number, but there are other subgroups of interest, such as instruments that will rotate the image by any angle or cross Fourier transform the two coordinates, that do not seem to have been considered in the literature. Finally, Section 7 offers a résumé and some closing comments.

2. STRUCTURE OF THE SYMPLECTIC GROUPS

Light rays in paraxial, three-dimensional geometric optics are characterized by their phase-space coordinates, written as a column vector $\mathbf{v} = (q_x, q_y, p_x, p_y)^\top$ referred to a Cartesian system (x, y) on a standard screen, whose normal at the origin is the optical axis, z. The position coordinates of the ray, $\mathbf{q} = (q_x, q_y)^\top \in \Re^2$, indicate its intersection with the screen, and the momentum or direction coordinates are $\mathbf{p} = (p_x, p_y)^\top \in \Re^2$. For a small neighborhood $|p| \ll 1$, p_x and p_y are the angles from the optical axis to the x and y coordinates of the ray times the refractive index of the medium. Beyond this neighborhood the geometric interpretation becomes invalid but, because of the simplicity of linear vector spaces, one adopts the extension of the range of momentum to the full plane; this is the paraxial model of geometric (and wave) optics. Mathematically, there is no reason to limit the position and momentum vectors to two components; we apply the following considerations for generic dimension N.

A. Linear Canonical Transformations

The Poisson bracket between two functions of phase space $\Re^{2N}$, $f(\mathbf{v})$ and $g(\mathbf{v})$, is[4]

$$\{f, g\} = \frac{\partial f}{\partial \mathbf{q}} \cdot \frac{\partial g}{\partial \mathbf{p}} - \frac{\partial f}{\partial \mathbf{p}} \cdot \frac{\partial g}{\partial \mathbf{q}} = \left(\frac{\partial f}{\partial \mathbf{q}}, \frac{\partial f}{\partial \mathbf{p}}\right) \boldsymbol{\Omega} \begin{pmatrix} \partial g/\partial \mathbf{q} \\ \partial g/\partial \mathbf{p} \end{pmatrix}, \tag{1}$$

where

$$\boldsymbol{\Omega} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \tag{2}$$

and allows us to write the Hamilton equations for evolution of rays along the optical axis as

$$\frac{\mathrm{d}\mathbf{v}(z)}{\mathrm{d}z} = \{H, \mathbf{v}(z)\}, \qquad H = \frac{\mathbf{p}^2}{2n_0} - \nu(\mathbf{q}, z), \tag{3}$$

where $n_0 + \nu(\mathbf{q}, z)$ is the refractive index of the medium and $H(\mathbf{v}, z)$ is the paraxial Hamiltonian function. The paraxial model of optics is thus an integrable Hamiltonian system, and $\mathbf{v} \in \Re^{2N}$ is its phase space. A transformation of phase space $\mathbf{v} \mapsto \mathbf{v}'(\mathbf{v})$ that maps a Hamiltonian system onto a Hamiltonian system is called canonical, and the Poisson brackets between the components of $\mathbf{v}'(\mathbf{v})$ are the same as between the components of the original $\mathbf{v}$, i.e., $\{v_i', v_j'\} = \Omega_{i,j} = \{v_i, v_j\}$. Canonical transformations are invertible, and we note that $\boldsymbol{\Omega}^\top = -\boldsymbol{\Omega}$ and $\boldsymbol{\Omega}^2 = -\mathbf{1}$.

We are interested here in linear canonical transformations $\mathcal{T}(\mathbf{M})$ of functions of phase space, where $\mathbf{M}$ is a matrix and $\mathbf{v} \mapsto \mathbf{M}\mathbf{v}$. When we follow a ray $\mathbf{v}$ as usual from left to right, which passes first through an optical element a given by a transformation $\mathcal{T}(\mathbf{M}_a)$ and second through an element b given by $\mathcal{T}(\mathbf{M}_b)$, then the map of the phase-space coordinates is

$$\begin{aligned} \mathbf{v} \mapsto \mathbf{v}' &= \mathbf{M}_a \mathbf{v}, \\ \mapsto \mathbf{v}'' &= \mathbf{M}_b \mathbf{v}' = \mathbf{M}_b \mathbf{M}_a \mathbf{v}. \end{aligned} \tag{4}$$

The composition rule for the abstract transformations $\mathcal{T}(\mathbf{M})$ that will follow the order of placement of the optical elements along the z axis is anti-isomorphic to the matrix product:

$$\mathcal{T}(\mathbf{M}_a)\mathcal{T}(\mathbf{M}_b) = \mathcal{T}(\mathbf{M}_b\mathbf{M}_a). \tag{5}$$

Here and in Section 3 we shall work with matrices. In Section 4, when optical systems are built, we shall find Eq. (5) useful.

The requirement that the components of $\mathbf{v}'(\mathbf{v})$ have the same Poisson brackets [Eq. (1)] as those of $\mathbf{v}$ leads to the symplectic conditions

$$\mathbf{M}\boldsymbol{\Omega}\mathbf{M}^\top = \boldsymbol{\Omega}. \tag{6}$$

Below, we shall show that there is only one connected set of solutions to Eq. (6), so

$$\det \mathbf{M} = 1. \tag{7}$$

The matrices that satisfy Eq. (6) are called symplectic matrices. It can immediately be seen that $\boldsymbol{\Omega}$ is symplectic and that, if $\mathbf{M}$ is symplectic, so are its transpose $\mathbf{M}^\top$ and its inverse $\mathbf{M}^{-1}$. Moreover, the product of two symplectic matrices is symplectic, unit matrix $\mathbf{1}$ is symplectic, the inverse $\mathbf{M}^{-1} = -\boldsymbol{\Omega}\mathbf{M}^\top\boldsymbol{\Omega}$ always exists and is symplectic, and associativity holds as it does for all matrices. The set of $2N \times 2N$ real symplectic matrices thus forms the group denoted $\mathrm{Sp}(2N, \Re)$. When $\mathbf{M}$ is written in 2×2 block form,

$$\mathbf{M} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}, \tag{8}$$

the symplecticity condition [Eq. (6)] reads as

$$\begin{bmatrix} \mathbf{A}\mathbf{B}^\top - \mathbf{B}\mathbf{A}^\top & \mathbf{A}\mathbf{D}^\top - \mathbf{B}\mathbf{C}^\top \\ \mathbf{C}\mathbf{B}^\top - \mathbf{D}\mathbf{A}^\top & \mathbf{C}\mathbf{D}^\top - \mathbf{D}\mathbf{C}^\top \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{9}$$

For 2×2 matrices this entails the single scalar restriction [Eq. (7)] on their four elements, so $\mathrm{Sp}(2, \Re)$ has three parameters. For 4×4 matrices, the antisymmetric matrix, Eq. (9), yields 6 independent bilinear equations among the 16 elements of the matrix, so $\mathrm{Sp}(4, \Re)$ has 10 free parameters. Correspondingly, the $2N \times 2N$ matrices of $\mathrm{Sp}(2N, \Re)$ that apply to $(N + 1)$-dimensional paraxial optics have $2N^2 + N$ independent, real parameters.

B. Iwasawa Decomposition of Sp(2N, ℜ)

We now analyze the manifold of symplectic matrices. To do this, Bargmann[2,3] applied a complex similarity transformation, for reasons much like those used in the study of multivalued real and matrix functions, to explain the two-fold cover that the spin provides over the orbital angular momentum. Indeed, the symplectic groups have in common with the complex logarithm function an infinity of Riemann sheets; but this is at best merely an analogy, which, nevertheless, we shall exploit to understand the rather peculiar manifold of the symplectic groups.

We consider here the generic Iwasawa (or *NAK*) decomposition[5,6] of an arbitrary, real nonsingular $2N \times 2N$ matrix $\mathbf{M}$ into the (unique and global) product $\mathbf{SR}$ of a solvable matrix $\mathbf{S}$ (lower-triangular, with positive elements on the diagonal) and an orthogonal matrix, $\mathbf{R}^{\top} = \mathbf{R}^{-1}$. Both sets of matrices are groups by themselves, and, since $\mathbf{S1}$ and $\mathbf{1R}$ are symplectic matrices, so are $\mathbf{S}$ and $\mathbf{R}$ themselves; thus $\mathbf{S\Omega S} = \mathbf{\Omega} = \mathbf{R\Omega R}^{-1}$. The orthogonal matrices constitute a maximal compact subgroup (K, of finite volume); the solvable part is itself the product of an Abelian subgroup (A, of mutually commuting elements) consisting of positive-definite diagonal matrices, and a nilpotent subgroup (N, of lower-triangular 2×2-block matrices, which have unit entries along the full diagonal, and where one $N \times N$ diagonal block is lower-triangular). Unlike for the elliptic subgroup K, powers of the matrices of the hyperbolic subgroup A and those of the parabolic subgroup N grow without bounds and never return to the identity (i.e., they are noncompact). Even though the defining symplecticity condition [Eq. (6)] may appear to permit the values ± 1 for the determinant of symplectic matrices, the value -1 can never occur because in the decomposition $\mathbf{M} = \mathbf{SR}$, $\det \mathbf{S} = 1$, and also $\det \mathbf{R} = 1$.

A $2N \times 2N$ solvable matrix [Eq. (8)] will have its upper-right block $\mathbf{B} = \mathbf{0}$, and hence $2N^2 + 2N$ parameters; but we see from Eq. (9) that $\mathbf{AD}^{\top} = \mathbf{1}$ and so $\mathbf{D}$ is completely determined by (solvable) $\mathbf{A}$; hence $\frac{1}{2}(N^2 + N)$ parameters are removed. Then, because $\mathbf{CD}^{\top}$ is symmetric, $\mathbf{C}$ is subject to another $\frac{1}{2}(N^2 - N)$ restrictions, leaving thus only $N^2 + N$ free parameters for the solvable matrix. The orthogonal matrix thus contains N^2 free parameters ranging in a compact domain: They lie in the intersection of the group $\mathrm{SO}(2N)$ of $2N$-dimensional real orthogonal matrices of unit determinant and $\mathrm{Sp}(2N, \mathfrak{R})$; they have the form of Eq. (9) with $\mathbf{C} = -\mathbf{B}$ and $\mathbf{D} = \mathbf{A}$, for which Eq. (6) reads as $(\mathbf{A} + i\mathbf{B})(\mathbf{A} - i\mathbf{B})^{\top} = 1$. This means that $\mathbf{A} + i\mathbf{B}$ are $N \times N$ unitary matrices $\mathrm{U}(N)$ but are written as $\mathbf{R}$ in a $2N \times 2N$ real form.[7] Finally, we realize that, whereas the manifold of solvable matrices is the Cartesian manifold $\mathfrak{R}^{N^2+N}$, the manifold of $\mathrm{U}(N)$ is the simply connected manifold of $\mathrm{SU}(N)$ (the group of $N \times N$ unitary matrices of unit determinant) times the circle of phases $\mathrm{U}(1)$, modulo the matrices $\exp(2\pi ik/N)\mathbf{1}$, for $k = 0, 1, 2, \ldots, N - 1$, that belong simultaneously to $\mathrm{SU}(N)$ and $\mathrm{U}(1)$; this quotient set, denoted Z_N, is all too often overlooked. Thus we conclude that, as a manifold, the symplectic group has the global structure[7]

$$\mathrm{Sp}(2N, \mathfrak{R}) \sim \mathfrak{R}^{N^2+N} \times [\mathrm{SU}(N) \times \mathrm{U}(1)]/Z_N. \quad (10)$$

C. Connectivity and Covers of $\mathrm{Sp}(2N, \mathfrak{R})$

The importance of this manifold decomposition is that it shows the nonsimple connectivity of the symplectic group manifold to be that of circle $\mathrm{U}(1)$. This means that $\mathrm{Sp}(2N, \mathfrak{R})$ is a connected, but infinitely connected, manifold. Unlike the group $\mathrm{SO}(3)$, which admits only $\mathrm{SU}(2)$ as the double and universal cover, $\mathrm{Sp}(2N, \mathfrak{R})$ can be covered any number of times. Its double cover is $\mathrm{Mp}(2N, \mathfrak{R})$, called the metaplectic group, which appears as the group of canonical[8] (or generalized Fresnel[9]) integral transforms, where $\mathrm{U}(1)$ is covered twice: As the 2π face of a clock is covered twice by one rotation of the Earth, so the clock angles should be counted modulo 4π. This symplectic–metaplectic connection is at the heart of the Maslov index and the Gouy phase. It may be remarked that the Gouy phase has received much attention in class and quantum optics in recent years[5,10]; it has been shown to be the geometric phase associated with the Lobachevskian or hyperbolic geometry that is inherently associated with the symplectic group.[11]

Similar three-fold, four-fold, etc. covers extend $\mathrm{U}(1)$, counting its angle modulo 6π, 8π, etc. The infinite cover of the circle is the real line that parameterizes the elements of the universal covering group $\overline{\mathrm{Sp}(2N, \mathfrak{R})}$. In other words, $\overline{\mathrm{Sp}(2N, \mathfrak{R})}$ has the manifold structure $\mathfrak{R}^{N^2+N+1} \times \mathrm{SU}(N)$. In the $N = 1$ case, as we shall see, $\mathrm{Sp}(2, \mathfrak{R})$ covers twice the radial paraxial group of 3×3 pseudo-orthogonal matrices $\mathrm{SO}(2, 1) = \mathrm{Sp}(2, \mathfrak{R})/\mathcal{Z}_2$; this fact has provided the basis for several applications in optics.[12,13] In the $N = 2$ case, it turns out that $\mathrm{Sp}(4, \mathfrak{R})$ covers twice the de Sitter group of $3 + 2$ pseudo-orthogonal matrices $\mathrm{SO}(3, 2) = \mathrm{Sp}(4, \mathfrak{R})/\mathcal{Z}_2$; this result has found applications in classic and quantum optics.[14] However, these connections (accidental homomorphisms) with the pseudo-orthogonal groups occur only in low dimensions.

3. $\mathrm{Sp}(2, \mathfrak{R})$ IN TWO-DIMENSIONAL PARAXIAL OPTICS

The previous argument on manifold connectivity was presented abstractly. For the reader to appreciate the solution when faced with the product of two $2N \times 2N$ symplectic matrices it will prove sufficient to follow the explicit Iwasawa product in the case $N = 1$ of two-dimensional paraxial optics. In relation (10), moreover, $\mathrm{SU}(1) = 1$ is the identity and the 2×2 real symplectic matrices satisfy Eq. (9) identically, so the only remaining condition is that the determinant be unity. In the Iwasawa decomposition parameters we indicate the matrices in the following way:

$$\begin{aligned}
\begin{bmatrix} a & b \\ c & d \end{bmatrix} &= \begin{bmatrix} e^{-\beta} & 0 \\ -\gamma e^{-\beta} & e^{\beta} \end{bmatrix} \begin{bmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{bmatrix} \\
&= \mathbf{M}_I(\gamma, \beta, \omega) \\
&= \mathbf{M}_I(\gamma, \beta, 0)\mathbf{M}_I(0, 0, \omega) \\
&= \mathbf{M}_I(\gamma, 0, 0)\mathbf{M}_I(0, \beta, 0)\mathbf{M}_I(0, 0, \omega), \\
&\qquad -\infty < \gamma, \quad \beta < \infty, \quad 0 \leqslant \omega < 2\pi, \quad (11)
\end{aligned}$$

and the relation to the *abcd* parameters is

$$e^{-\beta} = +\sqrt{a^2 + b^2} > 0,$$

$$\gamma = -\frac{ac + bd}{a^2 + b^2},$$

$$\omega = \arg(a + ib). \quad (12)$$

In the *abcd* parameters, the group composition law is simplest but the U(1) connectivity is invisible. When we use the Iwasawa parameters for the product of two such matrices, we find that

$$\begin{aligned}\mathbf{M}_I(\gamma, \beta, \omega) &= \mathbf{M}_I(\gamma_2, \beta_2, \omega_2)\mathbf{M}_I(\gamma_1, \beta_1, \omega_1)\\ &= \mathbf{M}_I(\gamma_2, \beta_2, 0)\mathbf{M}_I(0, 0, \omega_2)\mathbf{M}_I(\gamma_1, \beta_1, 0)\\ &\quad\times \mathbf{M}_I(0, 0, \omega_1)\\ &= \mathbf{M}_I(\gamma_2, \beta_2, 0)\mathbf{M}_I(\gamma_3, \beta_3, \omega_3)\mathbf{M}_I(0, 0, \omega_1)\\ &= \mathbf{M}_I(\gamma_2 + \gamma_3 e^{2\beta_2}, \beta_1 + \beta_3, \omega_3 + \omega_1),\end{aligned} \tag{13}$$

with the parameters (indicated by subscripts 3) of the middle matrix obtained from Eqs. (12) and given explicitly by

$$\begin{aligned}e^{-2\beta_3} &= (\cos\omega_2 e^{-\beta_1} - \sin\omega_2 e^{-\beta_1}\gamma_1)^2 + (\sin\omega_2 e^{\beta_1})^2,\\ \gamma_3 &= \cos 2\omega_2 e^{-2\beta_1}\gamma_1\\ &\quad + \tfrac{1}{2}\sin 2\omega_2[e^{-2\beta_1}(\gamma_1^2 - 1) + e^{2\beta_1}],\\ \omega_3 &= \arg(a_3 + ib_3)\\ &= \arg(\cos\omega_2 e^{-\beta_1} + \sin\omega_2 e^{-\beta_1}\gamma_1 + i\sin\omega_2 e^{\beta_1}).\end{aligned} \tag{14}$$

Since a and b cannot be simultaneously zero, the argument in Eqs. (14) is always well defined and can be compounded by Eq. (13) to any value $\bar{\omega} \in \Re$ to label uniquely the elements of the universal cover $\overline{\mathrm{Sp}(2,\Re)}$. If we work in $\mathrm{Sp}(2,\Re)$, we count $\bar{\omega}$ modulo 2π; if in $\mathrm{Mp}(2,\Re)$, modulo 4π. Generally we can record the phase by an integer winding number n_w so $\bar{\omega} = 2\pi n_w + \omega$, n_w counted modulo the cover of $\mathrm{Sp}(2,\Re)$ and angle ω counted modulo 2π times the cover. The full composition rule for the Iwasawa parameters is Eqs. (13) and (14), but for most practical cases we can use the *abcd* form. For the metaplectic group integral kernel, however, the two distinct elements $\mathbf{M}_I(\beta, \gamma, \omega)_{n_\omega=0}$ and $\mathbf{M}_I(\beta, \gamma, \omega)_{n_\omega=1} = \mathbf{M}_I(\beta, \gamma, \omega + 2\pi)$ have the same representative matrix and correspond to the same element of the symplectic group: $\mathrm{Mp}(2,\Re)$ and $\overline{\mathrm{Sp}(2,\Re)}$ have no faithful finite-dimensional matrix representatives.

As we shall now see, the Iwasawa decomposition builds optical systems with the basic blocks of imaging systems (composed of lenses and pure magnifiers, in NA) and fractional Fourier transformers (in K). Of course, lenses and free spaces only can also be used, but the resulting parameterization for the manifold of paraxial systems is inconvenient, as it separates the manifold into regions that are realizable by one, two, and three lenses;[15,16] from our point of view, this hides the simpler structure afforded by the Iwasawa decomposition.

4. TWO-DIMENSIONAL OPTICAL ELEMENTS

Propagation (displacement) through positive distances in free space, and thin lenses are the two elementary constituents of aligned, paraxial two-dimensional optical systems.

A. Displacements and Lenses

The action of a displacement on the ray and on phase space-coordinates is shown in Fig. 1. (We can equivalently see this as free propagation through empty space.) It is

$$\mathcal{D}(z) = \mathcal{T}[\mathbf{D}(z)], \qquad \mathbf{D}(z) = \begin{bmatrix} 1 & z\\ 0 & 1\end{bmatrix},$$

$$\mathcal{D}(z) : \begin{pmatrix} q\\ p\end{pmatrix} \mapsto \begin{pmatrix} q + zp\\ p\end{pmatrix}, \tag{15}$$

with $z \geqslant 0$ occurring naturally; note that $\mathcal{D}(z_1)\mathcal{D}(z_2) = \mathcal{D}(z_1+z_2)$.

Next, the map of rays and phase space that is due to a thin lens of Gaussian power g is shown in Fig. 2; it is

$$\mathcal{L}(g) = \mathcal{T}[\mathbf{L}(g)], \qquad \mathbf{L}(g) = \begin{bmatrix} 1 & 0\\ -g & 1\end{bmatrix},$$

$$\mathcal{L}(\mathbf{g}) : \begin{pmatrix} q\\ p\end{pmatrix} \mapsto \begin{pmatrix} q\\ p - gq\end{pmatrix}. \tag{16}$$

The Gaussian power g is the reciprocal of the focal length: $f = 1/g$; positive g means that the lens is convex (as in Fig. 2), and negative g indicates a concave lens. The set of thin-lens transformations [Eq. (16)] can be concatenated as $\mathcal{L}(g_1)\mathcal{L}(g_2) = \mathcal{L}(g_1+g_2)$, the neutral element (group unit) is $\mathcal{L}(0)$, the inverse of $\mathcal{L}(g)$ is $\mathcal{L}(-g)$, and associativity holds. These transformations constitute the one-parameter nilpotent Iwasawa subgroup N. Displacements (15) on the other hand, form only a *semi*group with identity, since z is physically restricted to the region $z \geqslant 0$. With optical arrangements built of displacements and thin lenses we can reach all other elements of $\mathrm{Sp}(2,\Re)$, including free propagation $\mathcal{D}(z)$ corresponding to *negative* values of z, as we now proceed to show.

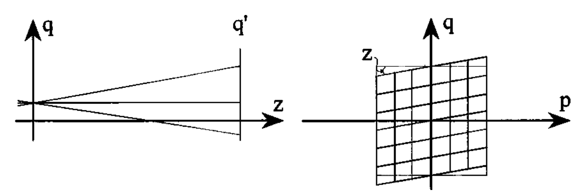

Fig. 1. Free displacement along the optical axis (left) acts on phase space by slanting the coordinate grid vertically (right); the map is $(q', p') = (q + zp, p)$. The ray angle ($\sim p$) and the area of phase space are conserved.

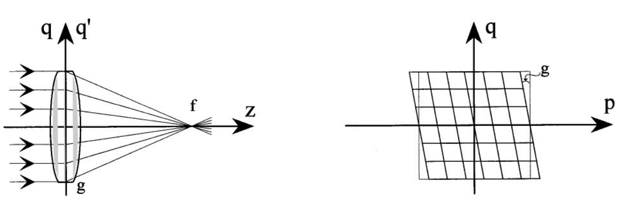

Fig. 2. A lens acts on an incoming bundle of rays (left) through horizontal slanting of the phase-space coordinate grid (right); the map is $(q', p') = (q, p - gq)$. A convex lens of Gaussian power $g > 0$ turns parallel rays ($p = 0$) to cross the z axis at focal distance $f = 1/g$. At the plane of the lens where the transformation takes place, the position q of the rays and the phase-space area are conserved.

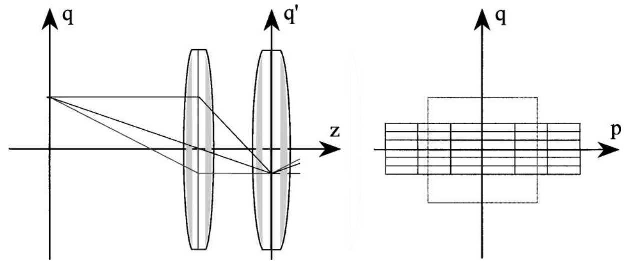

Fig. 3. Left, a pure magnifier built with two convex lenses in the DLDL configuration; right, in phase space, as the position coordinate is squeezed, the momentum coordinate is stretched to conserve areas. The magnifier is called pure because the concomitant slanting of phase space in a DLD configuration is corrected by the rightmost lens, which is coincident with the output screen.

B. Magnifiers

We concatenate two displacements and one lens in the following arrangement, labeled DLD:

$$\mathcal{D}(z_1)\mathcal{L}(g)\mathcal{D}(z_2) = \mathcal{T}\begin{bmatrix} 1 - z_2 g & z_1 + z_2 - z_1 g z_2 \\ -g & 1 - g z_1 \end{bmatrix}. \tag{17}$$

If $g = 1/z_1 + 1/z_2$ (called the focal condition), the upper-right element of Eq. (17) is zero. The arrangement is then an imaging system because the position of the image ray depends only on the position of the object ray; the map is $q \mapsto (1 - z_2 g)q$, with magnification factor $\zeta = 1 - z_2 g \in \Re$.

But Eq. (17) is not a *pure* magnifier because the momentum (angle) of the image ray is a linear combination of the object p and q and not of p only. For the magnifier to be pure, a final lens must be added, with a Gaussian power g' such that the lower-left element cancels, and the DLDL configuration is represented by a purely diagonal matrix. This happens when $g' = -g/(1 - z_2 g)$ and results in the phase-space map of Fig. 3. It is the magnifier

$$\mathcal{M}(\zeta) = \mathcal{D}(z_1)\mathcal{L}(g)\mathcal{D}(z_2)\mathcal{L}(g') = \mathcal{T}[\mathbf{M}(\zeta)]$$

$$\mathbf{M}(\zeta) = \begin{bmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{bmatrix}, \qquad \zeta = 1 - z_2 g. \tag{18}$$

For $0 < z_1, z_2 < \infty$, the focal condition implies that $z_1, z_2 > g^{-1}$, and hence $\zeta = 1 - z_2 g$ is negative; we can obtain positive magnification by concatenating two such systems. In this way we realize all diagonal matrices that are elements of the Iwasawa Abelian subgroup A. Of course, the DLD arrangements that satisfy the focal condition will by themselves also form the solvable group NA of lower-triangular matrices.

C. Phase-Space Rotators

The third Iwasawa subgroup, K, is the group of rotations of phase space, also called fractional Fourier transformations.[17] If we try to build a phase-space rotation in a single-lens configuration, $\mathcal{D}(z_1)\mathcal{L}(g)\mathcal{D}(z_2)$, we are forced to have $z_1 = z_2 = z$, as can be seen from Eq. (17). Thus we obtain the following single-lens realization of phase-space rotations shown in Fig. 4:

$$\mathcal{F}(\theta) = \mathcal{D}(z)\mathcal{L}(g)\mathcal{D}(z) = \mathcal{T}[\mathbf{F}(\theta)], \tag{19}$$

$$\mathbf{F}(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}; \tag{20}$$

$$g = \sin\theta > 0, \qquad z = \tan(\theta/2) > 0. \tag{21}$$

Thus $\mathcal{F}(\theta)$ for $0 < \theta < \pi$ can indeed be realized in the single-lens configuration involving a convex lens. The Fourier transform corresponds to $\theta = \pi/2$. By concatenating two such systems, $\mathcal{F}(\theta_1)\mathcal{F}(\theta_2) = \mathcal{F}(\theta_1 + \theta_2)$, we can realize the range $0 < \theta < 2\pi$. The element $\theta = 0 \equiv 2\pi$ corresponding to the identity cannot be realized in such a two-lens configuration (excluding the trivial possibility of having no lenses and no free flights). This is not the most general two-lens configuration, because the middle free flight is constrained to be the sum of the two free flights at the ends. It turns out that the situation with regard to the realization of the identity element does not change even when this constraint is released (see Subsection 4.E below).

Note that the matrix of $\mathcal{F}(\pi)$ is $\mathbf{F}(\pi) = -\mathbf{1}$, and hence commutes with all paraxial transformations, and that $\mathbf{F}(2\pi) = \mathbf{1}$ but with winding number $n_w = 1$. Phase-phase rotators are thus responsible for the occurrence of the metaplectic sign in the paraxial wave-optics integral transform when the angle θ exceeds 2π. This subgroup of maps can be also produced by a positive length of a graded-index waveguide with refractive-index profile $n(q) = n_0 - q^2$. The system is thus mathematically identical to the classic harmonic oscillator; its quantum or wave analog is the fractional Fourier transformation.[17]

D. Hyperbolic Expanders

The hyperbolic expander can be realized also with a single-lens configuration, provided that we use a concave lens as in Fig. 5. Then we have

$$\mathcal{H}(\zeta) = \mathcal{D}(z)\mathcal{L}(g)\mathcal{D}(z) = \mathcal{T}[\mathbf{H}(\zeta)],$$

$$\mathbf{H}(\zeta) = \begin{bmatrix} \cosh\zeta & \sinh\zeta \\ \sinh\zeta & \cosh\zeta \end{bmatrix}; \tag{22}$$

$$g = -\sinh\zeta < 0, \qquad z = \tanh\tfrac{1}{2}\zeta > 0. \tag{23}$$

The similarity between the realizations of rotators $\mathcal{F}(\theta)$ and hyperbolic expanders $\mathcal{H}(\zeta)$ is interesting. We can also propose as an optical element a length of hyperbolic waveguide with refractive-index profile $n(q) = n_0 + q^2$, which acts as a repulsive oscillator. Although $\mathcal{H}(\zeta)$ for $\zeta > 0$ is shown to be realizable with a single-lens configu-

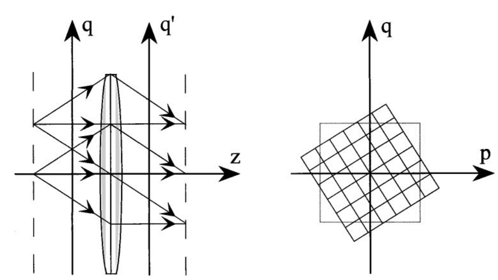

Fig. 4. Left, a fractional Fourier transformer built with a DLD arrangement; right, it rotates phase space. On the left, solid lines correspond to the object and image screens of the $\theta \sim 40°$ Fourier transformer whose action is shown on the right; dashed lines serve for the $\theta = \frac{1}{2}\pi$ Fourier transform.

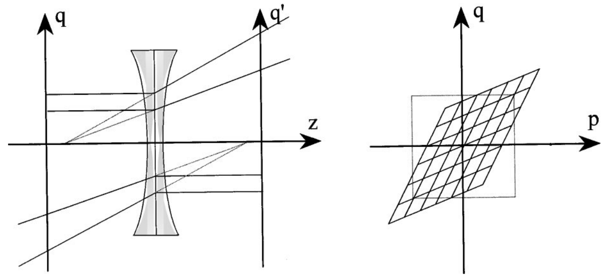

Fig. 5. Left, a hyperbolic expander built as DLD with one concave lens stretches phase space along a 45° line (right) and squeezes it along the orthogonal direction. A hyperbolic reducer cannot be built with a single lens.

ration, it turns out that the range $\zeta < 0$ cannot be realized, even in the two-lens configuration (see Subsection 4.E below).

The three basic group elements—lenses, magnifiers, and rotators—will produce every element of $\mathrm{Sp}(2,\mathfrak{R})$ when they are composed in the Iwasawa form [Eqs. (12)]. Conversely, every element of $\mathrm{Sp}(2,\mathfrak{R})$ can be decomposed into these optical elements.

E. Positive and Negative Ranges

A similarity transformation by rotators will intertwine between displacement and lens transformations. We may use the (realizable) rotators $\mathbf{M}_I(0,0,\frac{1}{2}\pi)_0$ and $\mathbf{M}_I(0,0,\frac{7}{2}\pi)_0 = \mathbf{M}_I(0,0,\frac{3}{2}\pi)_1 = \mathbf{M}_I(0,0,-\frac{1}{2}\pi)_0$, in the $\mathrm{Sp}(2,\mathfrak{R})$ double cover with the indicated winding numbers modulo 2 to compose

$$\mathcal{D}(g) = \mathcal{F}(\pm\tfrac{1}{2}\pi)\mathcal{L}(g)\mathcal{F}(\mp\tfrac{1}{2}\pi), \tag{24}$$

$$\mathcal{L}(z) = \mathcal{F}(\pm\tfrac{1}{2}\pi)\mathcal{D}(z)\mathcal{F}(\mp\tfrac{1}{2}\pi), \tag{25}$$

where we have taken advantage of the fact that $\mathcal{F}(\pi)$ commutes with all elements in $\mathrm{Sp}(2,\mathfrak{R})$. Equation (24) permits the construction of negative displacement transformations with paraxial optical arrangements.

A hyperbolic waveguide of negative length can be similarly built from one of positive length:

$$\mathcal{H}(-\theta) = \mathcal{F}(\pm\tfrac{1}{2}\pi)\mathcal{H}(\theta)\mathcal{F}(\mp\tfrac{1}{2}\pi). \tag{26}$$

A rotator $\mathbf{M}_I(0,0,\frac{1}{4}\pi)_0$ and its corresponding inverse element, $\mathbf{M}_I(0,0,\frac{15}{4}\pi)_0 = \mathbf{M}_I(0,0,\frac{7}{4})_1 = \mathbf{M}_I(0,0,-\frac{1}{4}\pi)_0$, can be used to turn a hyperbolic expander (with $\zeta \geqslant 0$) into a pure magnifier:

$$\mathcal{M}(\zeta) = \mathcal{F}(\tfrac{1}{4}\pi)\mathcal{H}(\zeta)\mathcal{F}(-\tfrac{1}{4}\pi). \tag{27}$$

F. Three Lenses Are Sufficient

We have demonstrated that the entire $\mathrm{Sp}(2,\mathfrak{R})$ group manifold of *abcd* matrices can be realized by use of thin lenses separated by free-propagation sections; however, one may ask what is the minimum number of lenses needed to realize a particular system. This question was thoroughly analyzed in Ref. 15. We quote only the principal results to wrap up our analysis:

1. Every $\mathrm{Sp}(2,\mathfrak{R})$ system can be realized in a configuration that involves no more than three lenses.

2. The region in the $\mathrm{Sp}(2,\mathfrak{R})$ manifold that cannot be realized in configurations that involve one or two lenses consists of the following two pieces:

$$\begin{bmatrix} a > 0 & b < 0 \\ c \leqslant 0 & d > 0 \end{bmatrix}, \qquad \begin{bmatrix} a > 0 & 0 \\ c \leqslant 0 & a^{-1} \end{bmatrix}. \tag{28}$$

We have already encountered in this section several examples of the three-parameter family in the first piece of the bad region [inequalities (28)], such as negative displacements $\mathcal{D}(z)$ with $z < 0$, noninverting hyperbolic expanders $\mathcal{H}(\zeta)$ with $\zeta > 0$, and pure, positive magnifiers $\mathcal{M}(\zeta)$ with $\zeta > 0$. These are examples of $\mathrm{Sp}(2,\mathfrak{R})$ systems that require three-lens configurations for their realization. The second piece in inequalities (28) is the two-parameter family of noninverting magnifiers, preceded or followed by the converging phase curvature of a convex lens. It describes that portion of the solvable part that cannot be realized in any configuration that involves fewer than three lenses. To realize the identity as a nontrivial concatenation of D's and L's requires that L occur in the arrangement a minimum of three times. (If the identity were realizable in the two-lens configuration, so also would be free flight through a negative distance.)

5. EXPONENTIAL-TYPE ELEMENTS OF $\mathrm{Sp}(2,\mathfrak{R})$ AND THEIR FRACTIONALIZATION

Equivalence relations between optical systems can be easily systematized by use of the following set related to the Pauli σ matrices

$$\boldsymbol{\tau}_1 = \boldsymbol{\sigma}_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

$$\boldsymbol{\tau}_2 = \boldsymbol{\sigma}_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$\boldsymbol{\tau}_0 = i\boldsymbol{\sigma}_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}; \tag{29}$$

$$\boldsymbol{\tau}_+ = \tfrac{1}{2}(\boldsymbol{\tau}_2 + \boldsymbol{\tau}_0) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

$$\boldsymbol{\tau}_- = \tfrac{1}{2}(\boldsymbol{\tau}_2 - \boldsymbol{\tau}_0) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}. \tag{30}$$

A. Strata of Matrices of the Exponential Type

Among the transformations seen in Section 4 [Eqs. (15), (16), (18), and (20)], the basic optical elements (positive length of a homogeneous medium or a thin lens) can be compounded easily. They are represented by exponential matrices as follows:

$$\mathbf{D}(z) = \exp(z\boldsymbol{\tau}_+), \qquad z \geqslant 0, \tag{31}$$

$$\mathbf{L}(g) = \exp(-g\boldsymbol{\tau}_-), \tag{32}$$

$$\mathbf{F}(\theta) = \exp(\theta\boldsymbol{\tau}_0), \qquad \theta \geqslant 0, \tag{33}$$

$$\mathbf{H}(\zeta) = \exp(\zeta\boldsymbol{\tau}_2), \qquad \zeta \geqslant 0. \tag{34}$$

Each of these elements can be fractionalized, i.e., built from two or more identical systems, each of which has a fraction of the parameter, e.g., $\mathcal{F}(\theta) = [\mathcal{F}(\theta/n)]^n$ or

$\mathcal{M}(\zeta) = [\mathcal{M}(\zeta/n)]^n$, because they lie upon one-parameter subgroups. It may come as a surprise, however, that not every paraxial optical system can be subjected to this process, contrary to what has been implied or stated occasionally in the literature.[18] This is so because $\mathrm{Sp}(2,\mathfrak{R})$ has one more commonly ignored peculiarity: It is not of the *exponential* type,[6,19] as we now proceed to put in evidence.

Matrices $\mathbf{M}$ of the exponential type are those for which a real matrix $\mathbf{T}$ exists such that $\mathbf{M} = \exp \mathbf{T}$. Symplectic matrices have unit determinants and, because $\det \mathbf{M} = \exp(\mathrm{tr}\,\mathbf{T})$, $\mathbf{T}$ must be traceless. In the 2×2 case, the matrices $\boldsymbol{\tau}_i$ in Eqs. (29) are a basis for all traceless matrices, so we can write

$$\mathbf{M}[x_1, x_2, x_0] = \exp(x_1\boldsymbol{\tau}_1 + x_2\boldsymbol{\tau}_2 + x_0\boldsymbol{\tau}_0) = \exp\begin{bmatrix} x_1 & x_2 + x_0 \\ x_2 - x_0 & -x_1 \end{bmatrix}, \quad (35)$$

indicating the polar parameters $\vec{x}$ of such $\mathbf{M}$ by brackets.

A similarity transformation of Eq. (35) by an $\mathrm{Sp}(2,\mathfrak{R})$ matrix will lead to a linear transformation of the polar parameter vector[12]:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\mathbf{M}[\vec{x}]\begin{bmatrix} a & b \\ \mathbf{c} & \mathbf{d} \end{bmatrix}^{-1} = \mathbf{M}[\vec{x}\,'], \quad (36)$$

$$\begin{pmatrix} x_1' \\ x_2' \\ x_0' \end{pmatrix} = \begin{bmatrix} ad + bc & cd - ab & -cd - ab \\ bd - ac & \frac{1}{2}(a^2 - b^2 - c^2 + d^2) & \frac{1}{2}(a^2 - b^2 + c^2 - d^2) \\ -bd - ac & \frac{1}{2}(a^2 + b^2 - c^2 - d^2) & \frac{1}{2}(a^2 + b^2 + c^2 + d^2) \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_0 \end{pmatrix}. \quad (37)$$

This leaves invariant the 2–1 norm of vector $\vec{x}$, namely,

$$-x_1^2 - x_2^2 + x_0^2 = \sigma\chi^2,$$
$$\sigma = +1 \ \text{(timelike) or}$$
$$\sigma = 0 \ \text{(lightlike) or}$$
$$\sigma = -1 \ \text{(spacelike)}. \quad (38)$$

The sign σ thus separates the exponential-type matrices [Eq. (35)] into three disjoint strata of one-parameter subgroups that have the following representatives:

$\sigma = +1$	elliptic:	rotator	$\mathbf{M}[0, 0, \theta]$,
$\sigma = 0$	parabolic:	displacement	$\mathbf{M}[0, -\frac{1}{2}z, \frac{1}{2}z]$,
		lens	$\mathbf{M}[0, \frac{1}{2}g, \frac{1}{2}g]$,
$\sigma = -1$	hyperbolic:	expander	$\mathbf{M}[0, -\zeta, 0]$,
		magnifier	$\mathbf{M}[-\zeta, 0, 0]$.

The determinants and the traces of matrices remain invariant under similarity transformations. This is true for the matrices $\mathbf{M}$ as well as for their logarithms (generators) $\mathbf{T}$. Now, since the matrices $\mathbf{M} \in \mathrm{Sp}(2,\mathfrak{R})$ have a unit determinant and the matrices $\mathbf{T}$ in its Lie algebra [Eq. (29)] have identically null trace, these two invariants are trivial. The determinant of the matrices in the algebra, Eq. (35), is the invariant norm [Eq. (36)]. There remains to be examined the trace of the symplectic matrices $\mathbf{M}$: since $\mathrm{tr}(\mathbf{M}'\mathbf{M}\mathbf{M}'^{-1}) = \mathrm{tr}\,\mathbf{M}$ for any $\mathbf{M}'$ in the group, it is sufficient to regard the representatives of the subgroup strata, Eqs. (16), (20), and (22). Excluding the group center $\{\mathbf{1}, -\mathbf{1}\}$, we thus divide the ranges of $\mathrm{tr}\,\mathbf{M} \in \mathfrak{R}$ into the following disjoint intervals:

$\sigma = +1$	elliptic:	$\mathrm{tr}\,\mathbf{M} = 2\cos\theta$,	$\in(-2, 2)$,
$\sigma = 0$	parabolic:	$\mathrm{tr}\,\mathbf{M} = +2$,	
$\sigma = -1$	hyperbolic:	$\mathrm{tr}\,\mathbf{M} = 2\cosh\zeta$,	$\in(2, \infty)$.

It should be appreciated that the two nontrivial invariants of symplectic $\mathbf{M} = \exp \mathbf{T}$ are not independent. From Eq. (35) they are related by

$$\mathrm{tr}\,\mathbf{M}[\vec{x}] = 2\cos\sqrt{\det \mathbf{T}[\vec{x}]}, \qquad \det \mathbf{T}[\vec{x}] = \sigma\chi^2, \quad (39)$$

as defined in Eq. (38).

B. Matrices of the Nonexponential Type

It is clear now that $\mathrm{Sp}(2,\mathfrak{R})$ matrices exist that are not accounted for in the previous enumeration; for example,

$$\begin{bmatrix} -1 & -z \\ 0 & -1 \end{bmatrix}, \qquad \begin{bmatrix} -\cosh\zeta & \sinh\zeta \\ \sinh\zeta & -\cosh\zeta \end{bmatrix}$$

do not belong to any one-parameter subgroup, because their traces are outside the range $(-2, \infty)$. Therefore, the group $\mathrm{Sp}(2,\mathfrak{R})$ is not of the exponential type. This fact plays an important role as an obstruction in the generalization of Hamilton's theory of turns, originally developed for the compact group SU(2),[20] to the noncompact group $\mathrm{SU}(1,1) = \mathrm{Sp}(2,\mathfrak{R})$.[19] This generalization is rather interesting because it so happens that SO(2, 1) *is* of the exponential type, whereas its covers $\mathrm{Sp}(2,\mathfrak{R})$, $\mathrm{Mp}(2,\mathfrak{R})$, and $\overline{\mathrm{Sp}(2,\mathfrak{R})}$ are not. The geometry of the one-parameter subgroups of these groups is analyzed in detail in Ref. 6. For our present purpose, however, it is sufficient to note that the region in the $\mathrm{Sp}(2,\mathfrak{R})$ manifold, through which no one-parameter subgroup passes, consists of all matrices $\mathbf{M}$ of the following two types:

$$\mathrm{tr}\,\mathbf{M} < -2, \ \text{or} \ \mathrm{tr}\,\mathbf{M} = -2 \ \text{except} \ \mathbf{M} = -1. \quad (40)$$

C. Fractionalization

For the fractionalization of a paraxial system $\mathcal{T}(\mathbf{M})$, the fact that $\mathrm{Sp}(2,\mathfrak{R})$ is not of the exponential type becomes an obstruction, and this subtle fact does not seem to have been appreciated (cf. Ref. 18). The fractionalization problem, however, can be solved for all matrices of the exponential type:

$$\mathbf{M}(\xi) = \exp(\xi\vec{u}\cdot\vec{\boldsymbol{\tau}}) = \begin{cases} \mathbf{1}\cos\xi + \vec{u}\cdot\vec{\boldsymbol{\tau}}\sin\xi & \mathrm{tr}\,\mathbf{M} = 2\cos\xi \in (-2,2) \\ \mathbf{1} + \xi\vec{u}\cdot\vec{\boldsymbol{\tau}} & \mathrm{tr}\,\mathbf{M} = 2 \\ \mathbf{1}\cosh\xi + \vec{u}\cdot\vec{\boldsymbol{\tau}}\sinh\xi & \mathrm{tr}\,\mathbf{M} = 2\cosh\xi > 2 \end{cases}, \tag{41}$$

where we denote by $\vec{u}$ a vector normalized to $|\chi| = 1$ in Eq. (38) for the elliptic and hyperbolic strata (and for the parabolic stratum we can agree that $u_0 = 1/2$). The generator matrix $\vec{u}\cdot\vec{\boldsymbol{\tau}}$ and the logarithm parameter ξ can be then found from the trace of Eq. (41), $m = \mathrm{tr}\,\mathbf{M}$ (and, for the parabolic subgroup, from its antidiagonal elements), as follows:

$$|m| < 2 \Rightarrow \vec{u}\cdot\vec{\boldsymbol{\tau}} = \frac{M - \frac{1}{2}m\mathbf{1}}{[1 - (\frac{1}{2}m)^2]^{1/2}},$$
$$\xi = \arccos\tfrac{1}{2}m + 2\pi n_w,$$
$$m = 2 \Rightarrow \vec{u}\cdot\vec{\boldsymbol{\tau}} = \frac{\mathbf{M} - \mathbf{1}}{M_{2,1} - M_{1,2}},$$
$$\xi = M_{2,1} - M_{1,2},$$
$$m > 2 \Rightarrow \vec{u}\cdot\vec{\boldsymbol{\tau}} = \frac{\mathbf{M} - \frac{1}{2}m\mathbf{1}}{[(\frac{1}{2}m)^2 - \mathbf{1}]^{1/2}},$$
$$\xi = \mathrm{arccosh}\,\tfrac{1}{2}m. \tag{42}$$

The fractionalization of an $\mathrm{Sp}(2,\mathfrak{R})$ matrix to an rth root $\mathbf{M}^{1/r}$ is thus solved by the same generator matrix $\vec{u}\cdot\vec{\boldsymbol{\tau}}$ and fractional parameter ξ/r. For the parabolic and hyperbolic cases the root is unique; in the elliptic case, however, there will be r roots distributed around the circle, $(\xi + 2\pi l)/r$, for $l = 0, 1, ...r - 1$. In the k-fold cover group of $\mathrm{Sp}(2,\mathfrak{R})$ the roots will be spaced by $2\pi k/r$. For $\mathrm{Sp}(2,\mathfrak{R})$ elements [relations (40)], which do not belong to any one-parameter subgroup, fractionalization cannot be defined in any sensible manner within $\mathrm{Sp}(2,\mathfrak{R})$. Our fractionalization procedure differs from that of Ref. 18, and, in particular, our procedure brings to light the existence of the nonexponential (nonfractionalizable) region of $\mathrm{Sp}(2,\mathfrak{R})$.

6. $\mathrm{Sp}(4,\mathfrak{R})$ IN THREE-DIMENSIONAL PARAXIAL OPTICS

A three-dimensional optical system transforms linearly the four-dimensional phase space of paraxial rays. The system is called axially symmetric when its elements are invariant under rotations about a common optical z axis and inversions through this axis; in this case the three-parameter group $\mathrm{Sp}(2,\mathfrak{R})$ is sufficient to identify all such systems. We consider now three-dimensional astigmatic (or nonaxially symmetric) systems, for which the understanding of the full ten-parameter group $\mathrm{Sp}(4,\mathfrak{R})$ is needed. The Cartan root diagram of the Lie algebra of $\mathrm{Sp}(4,\mathfrak{R})$ (Ref. 21) suggests that three well-chosen elements will be necessary and sufficient to produce the most general paraxial optical system for this dimension; we may use free displacements and two cylindrical lenses with distinct orientations. In this section we proceed systematically, examining in turn the nilpotent, Abelian, and compact subgroups of the NAK Iwasawa decomposition of $\mathrm{Sp}(4,\mathfrak{R})$. Their respective numbers of parameters are four, two, and four.

A. Displacements and Lenses

Free displacement in a homogeneous medium, $\mathcal{D}(z)$, is an axisymmetric optical element. The subgroup reduction $\mathrm{Sp}(4,\mathfrak{R}) \supset \mathrm{Sp}(2,\mathrm{R}) \times \mathrm{O}(2)$ contains the trivial representation of the rotation-and-inversion subgroup O(2) that rotates simultaneously the position and momentum $x{-}y$ planes. It is characterized by 4×4 matrices $\mathbf{D}(z)$ of the form of Eq. (15), with 2×2 unit matrix $\mathbf{1}$ in each block:

$$\mathbf{D}(z) = \begin{bmatrix} \mathbf{1} & z\mathbf{1} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}, \qquad z \geq 0. \tag{43}$$

Positivity is no real restriction since, as we saw in Eq. (24), we can invert the sign of z by means of spherical lenses.

An x-cylindrical lens of Gaussian power g_x has focal length $f_x = 1/g_x$ in the x direction; the generator axis of the cylinder is in the orthogonal y direction, where its power is $g_y = 0$. The lens transformation $\mathcal{L}_x(g_x)$ will map $(q_x, q_y, p_x, p_y)^{\mathsf{T}} \mapsto (q_x, q_y, p_x - g_x q_x, p_y)^{\mathsf{T}}$, as in Eq. (32). If the cylinder generator subtends an angle κ with the y axis, the representing 4×4 symplectic matrix [Eq. (8)] will have a symmetric lower-left block $\mathbf{R}(\kappa)\mathbf{C}\mathbf{R}(-\kappa)$, where $C = \mathrm{diag}(-g, 0)$ and $\mathbf{R}(\theta)$ is the rotation matrix [Eq. (20)]:

$$\mathbf{L}(g,\kappa) = \begin{bmatrix} & \mathbf{1} & & \mathbf{0} \\ -g\cos^2\kappa & g\cos\kappa\sin\kappa & & \\ & & & \mathbf{1} \\ g\cos\kappa\sin\kappa & -g\cos^2\kappa & & \end{bmatrix}. \tag{44}$$

Generally, two such cylindrical thin lenses, superposed and with different orientations, constitute the most general astigmatic thin-lens matrix:

$$\mathbf{L}(\mathbf{g}) = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ -\mathbf{g} & \mathbf{1} \end{bmatrix}, \qquad \mathbf{g} = \begin{bmatrix} g_{xx} & g_{xy} \\ g_{xy} & g_{yy} \end{bmatrix}. \tag{45}$$

Astigmatic lenses [Eq. (45)] represent elements of a three-parameter subgroup of lower-triangular block matrices. Spherical lenses are built of two cylindrical lenses with orthogonal generators and equal power g, so that the lower-diagonal block is $\mathbf{g} = g\mathbf{1}$, a multiple of the unit 2×2 matrix. Note that this three-parameter manifold does not yet exhaust the Iwasawa nilpotent subgroup N, which has four parameters. The parameter that we have missed is in the lower-left position of the **A** block and [because of the symplectic condition $\mathbf{AD}^{\top} = \mathbf{1}$ (Ref. 7)] also in the upper-right position of the **D** block. This extra parameter will now be accounted for.

B. Astigmatic Magnifiers

From Subsection 4.B we see that pure inverting magnifiers can be built with DLDL configurations. Consider one such magnifier along the x direction, built with cylindrical lenses of powers g_x and g_x' and (axisymmetric) displacements z_1 and z_2 such that $g_x = 1/z_1 + 1/z_2$ and $g_x' = -g_x/(1 - z_2 g_x)$ in the x entries, as given in Eq. (18), and the accompanying free displacement by $z_1 + z_2$ in the y entries. Now we build a second such magnifier along the y direction, with corresponding Gaussian powers g_y and g_y' and distances z_1' and z_2', placing it within the same available total length $z_1 + z_2 = z_1' + z_2'$. See Fig. 6. These systems $\mathcal{M}(z_1, z_2, z_1')$ are represented by negative-definite diagonal matrices:

$$\mathbf{M}(z_1, z_2, z_1') = \mathrm{diag}(1 - z_2 g_x, \mathbf{1} - z_2' g_y, \mathbf{1} - z_1 g_x, \mathbf{1} - z_1' g_y). \tag{46}$$

By concatenation with an axisymmetric image inverter, this diagonal matrix can be turned into a positive-definite matrix.

The set of astigmatic magnifiers [Eq. (46)] is a three-dimensional group, with parameters z_1, z_2, and z_1', say. As we shall now show, two of these parameters belong to Iwasawa Abelian group A, whereas the third is the nilpotent group parameter that we missed in subsection 6.A. Note first that the rotation of the axis of a cylindrical lens in the $x-y$ plane by an angle κ as in Eq. (44) will turn diagonal submatrices into symmetric submatrices. So, if the two cylindrical subsystems in Fig. 6 are set at an angle $\kappa \neq \pi/2$ to each other, they will not produce any extra freedom, because the symmetric submatrices **A** and **D** (satisfying $\mathbf{AD}^{\top} = \mathbf{1}$) can always be brought to diagonal form by a rotation in the $x-y$ plane to principal axes. However, if instead we now *factor* a rotation to the right, using the Iwasawa decomposition for the **A** block (and now keeping this rotation for Subsection 6.C), the matrix that remains of **A** is lower-triangular. The off-diagonal matrix element is thus the missing parameter of the nilpotent Iwasawa subgroup N seen in Subsection 6.A. The two remaining diagonal elements parameterize the Abelian subgroup A. Together, matrices in the Abelian and nilpotent subgroups of $\mathrm{Sp}(4, \mathfrak{R})$ constitute a six-parameter solvable group whose elements can be realized by optical arrangements such as that shown in Fig. 6, with the two cylindrical lenses at the exit having arbitrary power and orientation.

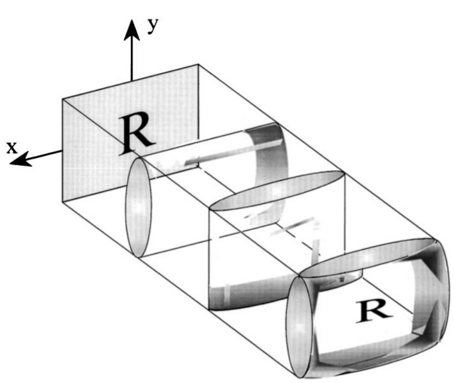

Fig. 6. Astigmatic magnifier built with one DLDL configuration in the x direction and another in the y direction. The two share the same input and output screens. The system is a pure astigmatic magnifier when the two superposed cylindrical lenses at the exit face (an astigmatic lens) are chosen appropriately. With all lens powers and orientations, these arrangements are all negative-definite elements of the solvable Iwasawa subgroup NA of $\mathrm{Sp}(4, \mathfrak{R})$; the positive-definite elements are obtained by the concatenation of two astigmatic magnifiers.

C. Gyrators of Phase Space

The pending subgroup in the Iwasawa decomposition is K, the four-parameter compact subgroup U(2) whose elements we shall generically call gyrators. They rotate four-dimensional phase space symplectically, including rotation of the image or the fractional Fourier transforms or both among the two pairs of canonically conjugate coordinates. Their central (commuting) subgroup will be shown here to be the set of axisymmetric fractional Fourier transformers.

Consider first the spherical-lens arrangement that rotates the x and y phase-space planes jointly; this is the direct generalization of the fractional Fourier transformers in Eq. (20) to 4×4 matrices with 2×2 blocks that are multiples of the identity. From the discussion preceding relation (10), the 2×2 unitary matrices $\mathbf{A} + i\mathbf{B}$ that correspond to these orthogonal transformations are

$$\begin{bmatrix} \cos\theta & 0 \\ 0 & \cos\theta \end{bmatrix} + i \begin{bmatrix} -\sin\theta & 0 \\ 0 & -\sin\theta \end{bmatrix} = e^{-i\theta}\mathbf{1} \in U(1). \tag{47}$$

Since they commute with all other unitary matrices $\mathbf{A}' + i\mathbf{B}'$, they are in the U(1) center of U(2) [although they do *not* commute with the rest of the $\mathrm{Sp}(4, \mathfrak{R})$ group]. Hence the corresponding 4×4 matrices

$$\mathbf{F}(\theta) = \begin{bmatrix} \cos\theta\mathbf{1} & \sin\theta\mathbf{1} \\ -\sin\theta\mathbf{1} & \cos\theta\mathbf{1} \end{bmatrix} \tag{48}$$

constitute the central U(1) submanifold of the compact Iwasawa subgroup. This subgroup carries the onus of the connectivity of $\mathrm{Sp}(4, \mathfrak{R})$ as well as its winding number.

To illustrate that by means of cylindrical lenses and free flights we can build any element in the compact submanifold $\mathrm{SU}(2) \subset \mathrm{Sp}(4, \mathfrak{R})$ in relation (10), we now examine two particularly important arrangements that we have not hitherto found described in the literature: image gyrators and cross gyrators.

D. Image Gyrators and Reflectors

An image gyrator $\mathcal{G}(\phi) = \mathcal{T}[\mathbf{G}(\phi)]$ rotates the position and momentum planes jointly by an angle ϕ. The desired effect of this arrangement is shown in Fig. 7 (and the system is *not* supposed to be produced with mirrors!). It is represented by the matrix whose block form is

$$\mathbf{G}(\phi) = \begin{bmatrix} \cos\phi & \sin\phi & \multicolumn{2}{c}{\mathbf{0}} \\ -\sin\phi & \cos\phi & \cos\phi & \sin\phi \\ \multicolumn{2}{c}{\mathbf{0}} & -\sin\phi & \cos\phi \end{bmatrix} = \begin{bmatrix} \cos\phi\mathbf{1} + \sin\phi\boldsymbol{\tau}_0 & \mathbf{0} \\ \mathbf{0} & \cos\phi\mathbf{1} + \sin\phi\boldsymbol{\tau}_0 \end{bmatrix}. \quad (49)$$

This gyrator can made out of two identical reflectors $\mathcal{I}$ placed at an angle. Figure 8 shows the reflector that we now proceed to construct.

Consider the transformation $\mathcal{J}(f)$ between an object and its inverted image, of unit magnification, produced by a convex cylindrical lens [written first as an $\mathrm{Sp}(2,\mathfrak{R})$ transformation] of focal distance $f = 1/g$:

$$\mathcal{J}(f) = \mathcal{D}(2f)\mathcal{L}(1/f)\mathcal{D}(2f) = \mathcal{T}\begin{bmatrix} -1 & 0 \\ -1/f & -1 \end{bmatrix}, \quad (50)$$

and the concatenation of two such inverting imagers:

$$[\mathcal{J}(f)]^2 = \mathcal{T}\begin{bmatrix} 1 & 0 \\ 2/f & 1 \end{bmatrix}. \quad (51)$$

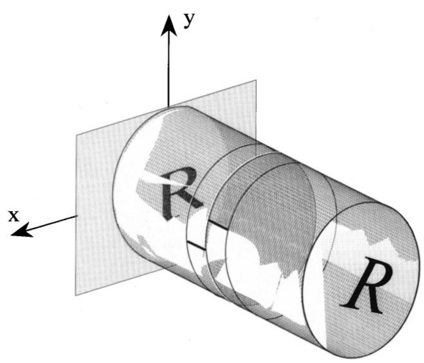

Fig. 7. An image gyrator is a paraxial instrument that will rotate the x–y planes of phase space. Does it exist? We can build it with two identical pieces (see Fig. 8) rotated at half the gyration angle.

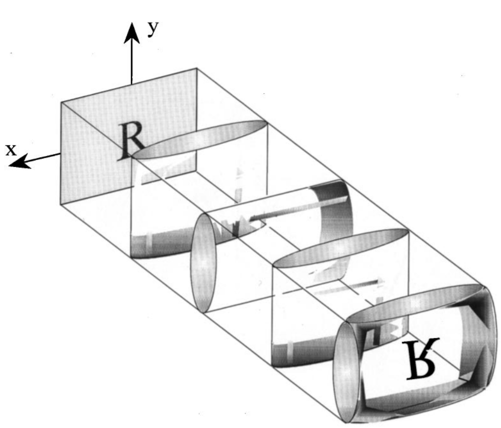

Fig. 8. The reflector preserves the x axis and inverts the y axis. It is built with inverting DLD subunits (cf. the two-dimensional Fig. 3), two for the x axis and (within the same available length) one on the y axis; at the end there is a common astigmatic lens to correct the slant of four-dimensional phase space.

The placement of a convex lens of focal distance f after arrangement (50) and of a convex lens of focal distance $f/2$ after two such arrangements [Eq. (51)] will yield the reflection (of winding number 0) and the unit (of winding number 1), respectively:

$$\mathcal{J}(f)\mathcal{L}(1/f) = \mathcal{T}\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad (52)$$

$$[\mathcal{J}(f)]^2\mathcal{L}(2/f) = \mathcal{T}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \quad (53)$$

Now we build the arrangement of Fig. 8 with two inverting imagers [Eq. (50)] in the x direction, indicated by $\mathcal{J}_x(f)$, and one inverting imager in the y direction, $\mathcal{J}_y(2f)$, of double focal length $2f$, finally correcting both for the diverging phase curvatures by means of an appropriate astigmatic lens. Thus we obtain the transformation represented by an $\mathrm{Sp}(4,\mathfrak{R})$ diagonal matrix:

$$\mathcal{I}_0 = \mathcal{J}_x(f)\mathcal{J}_y(2f)\mathcal{J}_x(f)\mathcal{L}_x(2/f)\mathcal{L}_y(\tfrac{1}{2}f) = \mathcal{T}(\mathbf{I}_0) = \begin{bmatrix} \mathcal{D}(2f)\mathcal{L}_x(1/f)\mathcal{D}(4f)\mathcal{L}_x(1/f)\mathcal{D}(2f)\mathcal{L}_x(2/f) \\ \mathcal{D}(4f)\mathcal{L}_y(1/2f)\mathcal{D}(4f)\mathcal{L}_y(1/2f) \end{bmatrix}, \quad (54)$$

$$\mathbf{I}_0 = \mathrm{diag}(1, -1, 1, -1). \quad (55)$$

In the second line of Eq. (54) the array has the evident vectorial meaning of the x and y components of the transformation. This is a reflector that inverts the y axis and is the unit (of winding number 1) in the x direction. Note that from matrices (49) and (55) it follows that $\mathbf{I}_0\mathbf{G}(\phi) = \mathbf{G}(-\phi)\mathbf{I}_0$.

Whereas Eq. (55) represents a reflection across a mirror placed on the $y = 0$ plane, one can obtain a mirror $\mathcal{I}_\phi$ at any other angle ϕ with the y axis by simply rotating the entire arrangement. For the matrices, we have

$$\mathbf{I}_\phi = \mathbf{G}(\phi)\mathbf{I}_0\mathbf{G}(-\phi). \quad (56)$$

Because the product of two reflections is a rotation, when we follow one reflector $\mathcal{I}_0$ with another at an angle $\frac{1}{2}\phi$, the result is the gyration (rotation) of angle ϕ between the x and y components of phase space:

$$\mathcal{G}(\phi) = \mathcal{I}_{\phi_0 + (1/2)\phi}\mathcal{I}_{\phi_0}. \quad (57)$$

This is the desired gyrator. It is an $\mathrm{Sp}(4,\mathfrak{R})$ transformation completely contained in the SU(2) subgroup, of winding number 0, that will yield an image identical to the object but rotated by the angle π. Clearly, placing the two reflectors at angles $\frac{1}{2}\phi = 0$ or π yields the same, perfect imager. Changing the angle between the two reflectors involves rotating only one reflector with respect to the other on the same axis, so an actual optical device with this property can be easily fabricated and adjusted.

Other means of building an image gyrator exist; for instance, use of a pair of dove prisms. Our point here is that image rotators and reflectors in planes that contain the optical z axis are elements of the symplectic group and as a matter of principle can be realized by use of only thin lenses and free displacements; this is what we have demonstrated here. The image gyrator is a system that is invariant under rotations about the optical axis, as attested to by the independence of Eq. (57) from ϕ_0; it is not fully axisymmetric, however, because a mirror-reflected version of the arrangement will rotate the image by $-\phi$.

E. Cross Gyrators

The cross gyrator is an optical element $\mathcal{X}_0(\phi) = \mathcal{T}[\mathbf{X}(\phi)]$ that performs joint rotations in the (q_x, p_y) and (q_y, p_x) phase spaces. It has the following matrix representation:

$$\mathbf{X}(\phi) = \begin{bmatrix} \cos\phi & 0 & 0 & \sin\phi \\ 0 & \cos\phi & \sin\phi & 0 \\ 0 & -\sin\phi & \cos\phi & 0 \\ -\sin\phi & 0 & 0 & \cos\phi \end{bmatrix} = \begin{bmatrix} \cos\phi\mathbf{1} & \sin\phi\boldsymbol{\tau}_2 \\ \sin\phi\boldsymbol{\tau}_2 & \cos\phi\mathbf{1} \end{bmatrix}, \tag{58}$$

where the latter expression uses the $\boldsymbol{\tau}_i$ matrices of Eqs. (29). We now show that the cross gyrator can be produced from the image gyrator $\mathcal{G}(\phi)$ defined in Eq. (49) through a similarity transformation by the phase-space rotator $\mathcal{F}_y(\frac{1}{2}\pi)$. This last element is a Fourier transformer [Eq. (20)] in the (q_y, p_y) phase-space plane and the identity transformation in the (q_x, p_x) plane; it can be built within the same length along the optical axis, as shown in Fig. 9, and represented by [cf. Eq. (20)]

$$\mathcal{F}_y(\tfrac{1}{2}\pi) = \begin{bmatrix} \mathcal{D}(\frac{1}{2})\mathcal{L}_x(4)\mathcal{D}(1)\mathcal{L}_x(4)\mathcal{D}(\frac{1}{2})\mathcal{L}_x(8) \\ \mathcal{D}(1)\mathcal{L}_y(1)\mathcal{D}(1) \end{bmatrix}$$

$$\mathbf{F}_y(\tfrac{1}{2}\pi) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \tag{59}$$

where again we use vector notation for the x and y directions. We note that the transformation $\mathcal{F}_y(\frac{1}{2}\pi)$ was used by Lohmann and co-workers[22] in an equivalent arrangement to produce optically a (smoothed or squared) Wigner function of a one-dimensional signal. The desired cross gyrator [Eq. (58)] is now obtained as

$$\mathcal{X}(\phi) = \mathcal{F}_y(\tfrac{1}{2}\pi)\mathcal{G}(\phi)\mathcal{F}_y(-\tfrac{1}{2}\pi) \tag{60}$$

and is shown in Fig. 9. One obtains image gyration simply by rotating the second arrangement together with the coordinate axes of the end screen.

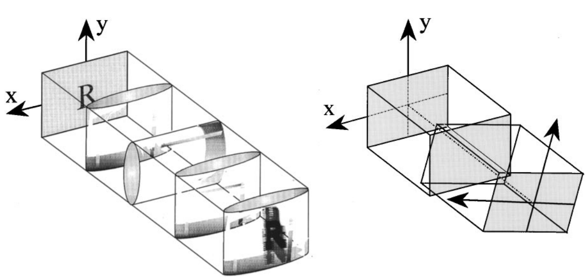

Fig. 9. Left, arrangement of a Fourier transformer in the y direction with perfect imaging in the x direction. On the right: Two such arrangements concatenated at an angle will produce cross gyration. Cross gyrators performs fractional Fourier transformations in the q_x–p_y and q_y–p_x planes.

Cross gyration for the angle $\phi = \frac{1}{4}\pi$ was used by Simon and Mukunda[23] to design a lens system that is capable of converting a familiar beam of a well-defined type into the twisted Gaussian Schell-model beam. This beam carries a novel type of nonseparable phase with definite chirality (handedness) that has come to be known as the twist phase. In an interesting subsequent work Friberg and collaborators[24] used the same lens system for an experimental realization of twisted Gaussian Schell-model beams.

F. Gyrators of U(2)

Performing a similarity transformation with the gyrator on the cross gyrator, we define

$$\mathcal{X}_\gamma(\alpha) = \mathcal{G}(\tfrac{1}{2}\gamma)\mathcal{X}(\alpha)\mathcal{G}(-\tfrac{1}{2}\gamma) \tag{61}$$

to obtain a family of rotated one-parameter subgroups of cross gyrators. The value $\gamma = \frac{1}{2}\pi$ corresponds to an important one-parameter subgroup for our purposes, so we single it out by denoting it through

$$\mathcal{Y}(\beta) = \mathcal{X}_{\pi/2}(\beta) = \mathcal{G}(\tfrac{1}{4}\pi)\mathcal{X}(\beta)\mathcal{G}(-\tfrac{1}{4}\pi) = \mathcal{T}[\mathbf{Y}(\beta)], \tag{62}$$

$$\mathbf{Y}(\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta & 0 \\ 0 & \cos\beta & 0 & -\sin\beta \\ -\sin\beta & 0 & \cos\beta & 0 \\ 0 & \sin\beta & 0 & \cos\beta \end{bmatrix} = \begin{bmatrix} \cos\beta\mathbf{1} & \sin\beta\boldsymbol{\tau}_1 \\ -\sin\beta\boldsymbol{\tau}_1 & \cos\beta\mathbf{1} \end{bmatrix}. \tag{63}$$

We can now incorporate the previous four transformations, Eqs. (48), (49), (58), and (63), into the structure of compact subgroup U(2) of Sp(4, $\Re$) and write them in a form that is surely familiar to the reader:

$$\begin{aligned} \mathcal{U}_0(\omega) &= \mathcal{F}(\omega) = \exp(-i\omega J_0) \leftrightarrow \exp(-i\omega\mathbf{1}), \\ \mathcal{U}_1(\alpha) &= \mathcal{X}(\alpha) = \exp(-i\alpha J_1) \leftrightarrow \exp(-i\alpha\boldsymbol{\sigma}_1), \\ \mathcal{U}_2(\beta) &= \mathcal{Y}(\beta) = \exp(-i\beta J_2) \leftrightarrow \exp(-i\beta\boldsymbol{\sigma}_2), \\ \mathcal{U}_3(\gamma) &= \mathcal{G}(\gamma) = \exp(-i\gamma J_3) \leftrightarrow \exp(-i\gamma\boldsymbol{\sigma}_3). \end{aligned} \tag{64}$$

The 4×4 generator matrices $\mathbf{J}_\mu$, $\mu = 0, 1, 2, 3$, defined through expressions (64), are readily found to have the following suggestive direct product forms:

$$\begin{aligned} \mathbf{J}_0 &= -\boldsymbol{\sigma}_2 \otimes \mathbf{1}, & \mathbf{J}_1 &= -\boldsymbol{\sigma}_2 \otimes \boldsymbol{\sigma}_1, \\ \mathbf{J}_2 &= -\boldsymbol{\sigma}_2 \otimes \boldsymbol{\sigma}_3, & \mathbf{J}_3 &= -\mathbf{1} \otimes \boldsymbol{\sigma}_2. \end{aligned} \tag{65}$$

The $\mathbf{J}$'s obey the same commutation relations as the 2×2 Pauli $\boldsymbol{\sigma}$'s (with $\boldsymbol{\sigma}_0 = \mathbf{1}$), the better-known form of the generators in the defining representation of the group U(2), namely,

$$[\mathbf{J}_j, \mathbf{J}_k] = 2i\epsilon_{jkl}\mathbf{J}_l, \qquad [\mathbf{J}_0, \mathbf{J}_k] = 0, \qquad j, k = 1, 2, 3. \tag{66}$$

G. Realizability and Fractionalizability

The one-parameter subgroup U(1) generated by J_0 and the three one-parameter subgroups of SU(2) generated by J_1, J_2, and J_3, as shown in expressions (64), can be concatenated to exhaust the Iwasawa maximal compact subgroup (K) of $\mathrm{Sp}(4,\Re)$, which is thus realizable by paraxial optical arrangements. [Indeed, we have done more work than necessary to establish the realizability of U(2); for instance, the two one-parameter subgroups generated by J_3 and J_1 imply the realizability of the whole SU(2) manifold.] Since the realizability of the Abelian (A) and nilponent (N) subgroups is already established, the realizability of the entire $\mathrm{Sp}(4,\Re)$ manifold of first-order optical systems with thin lenses follows from the global nature of the Iwasawa decomposition.[1]

The analysis of the ten-parameter $\mathrm{Sp}(4,\Re)$ for questions of minimal realizability is more arduous than that for $\mathrm{Sp}(2,\Re)$ undertaken in Section 5; it will not be presented here beyond Eqs. (65). In fact, it is through the homomorphic group SO(3, 2) that we can examine best the orbit structure as we did in Eq. (35), but now it is a five-dimensional space with metric $(+\,+\,-\,-\,-)$. As we proceeded there, however, the trace of the 4×4 representing matrices can give us the crucial information on whether a given group element is of the exponential type. Elements in the Iwasawa maximal compact group [see Eqs. (48), (49), (58), and (63)] have traces in the interval $[-4,4]$; we exclude the end values that occur for $-\mathbf{1}$ and $\mathbf{1}$. The Iwasawa nilpotent group is of exponential type by itself and is represented by matrices with 1's on the diagonal, so their trace is $+4$. The onus of the argument is again on the Iwasawa Abelian subgroup obtained from the astigmatic pure magnifiers; cf. Eqs. (18) and (46): Exponential-type matrices are of the form $\mathrm{diag}(\zeta_x, \zeta_y, \zeta_x^{-1}, \zeta_y^{-1})$, so their trace is a value in $[4,\infty)$. We conclude, as for inequality (40), that when a 4×4 symplectic matrix satisfies

$$\mathrm{tr}\,\mathbf{M} < -4 \quad \text{or} \quad \mathrm{tr}\,\mathbf{M} = -4 \quad \text{except} \quad \mathbf{M} = -\mathbf{1}, \tag{67}$$

it is not of the exponential type. The corresponding optical system then cannot be fractionalized.

H. SU(2) in Polarization Optics

To conclude this section we draw from polarization optics an analogy that throws additional light on the unitary group structure that appears in paraxial optical transformations. Polarization optics, too, is governed by an SU(2) structure,[25] with optically active media serving as image gyrators $\mathcal{G}(\gamma)$ and birefringent media acting as cross-gyrators $\mathcal{X}_\theta(\alpha)$. In particular, $\mathcal{X}_0(\frac{1}{4}\pi)$ and $\mathcal{X}_0(\frac{1}{2}\pi)$ correspond, respectively, to quarter-wave plates and half-wave plates, and θ is the angle of these plates about the propagation axis of the light beam.

On the basis of the Hamilton theory of turns[20] (see also Refs. 19 and 25), it was shown recently that every SU(2) transformation can be realized by use of two quarter-wave plates and one half-wave plate.[26] Transcribed to the paraxial optics context, this result says that every rotation of phase space $\mathcal{U}\in\mathrm{SU}(2)$ can be realized as

$$\mathcal{U}(\theta_1, \theta_2, \theta_3) = \mathcal{X}_{\theta_1}(\tfrac{1}{2}\pi)\mathcal{X}_{\theta_2}(\tfrac{1}{4}\pi)\mathcal{X}_{\theta_3}(\tfrac{1}{4}\pi), \tag{68}$$

where θ_1, θ_2, and θ_3 are well-defined linear combinations of the Euler parameters of SU(2).

This result gives a special status to $\frac{1}{2}\pi$ and $\frac{1}{4}\pi$ cross gyrators, which we may call quarter and half cross gyrators. The realizability of these cross gyrators amounts to that of the entire SU(2) manifold. From a practical point of view it is important that there is no need to change free-flight distances and focal powers of the lenses to go from the realization of one SU(2) element to that of another. Imagine an arrangement in which a half cross gyrator and two quarter cross gyrators are mounted coaxially, with circular dial rings attached to monitor their angular positions about the common axis. To realize a specific SU(2) transformation, one has only to rotate the three pieces about the common axis so the dials bear the values of the Euler parameters of the SU(2) element under consideration. It should be appreciated that using this gadget to produce this three-parameter group of transformations is as simple as anything can be.

7. CONCLUDING REMARKS

Weyl[27] designated the Cartan C family of semisimple groups "symplectic," using the Greek verb $+\pi\lambda\acute{\epsilon}\kappa\epsilon\iota\nu$, whose meaning is "to twine, plait, weave," to reflect their imbricate structure. Next to the Heisenberg–Weyl groups, the symplectic groups lie at the very root of paraxial geometric and wave optics. They are also the dynamic groups of the quantum-harmonic oscillators; thus they have been inherited by quantum optics to describe squeezed light and other phenomena of quantized fields. Here we have realized the transformations of the symplectic groups of two and four dimensions by classical geometric models of optical arrangements. This treatment applies practically *verbatim* to paraxial wave optics and provides the foundation for higher-order aberration geometric optics. For quantum optics, phase space can be seen through the Wigner function, as we indicate below.

The model of paraxial optical systems for monochromatic wave fields uses the same transformations of the symplectic groups but represented by integral transform kernels (the optical transfer functions of paraxial systems) that faithfully follow the metaplectic cover groups on Hilbert spaces of functions. Much work was done on this subject from the point of view of quantum mechanics[8,28] before it was recognized that it applied straightforwardly to optical models.[9,29] The results that we have presented here resolve the vexing problem of the metaplectic phase on the level of classical geometry; we deem this derivation to be clearer than the more arduous analytic properties of sign changes in Gaussian integrals that bear the *abcd* parameters. This group of linear homogeneous transformations can be extended, moreover, by adding the phase-space translations generated by q_j and p_k, representing a thin prism (which translates the ray angle, i.e., momentum) and a thin inclined slab (which translates the ray position), respectively; the extended paraxial group is called Weyl symplectic.[29]

The generators of the symplectic groups have also been used as a foundation for the theory of aberrations in metaxial geometric optics.[30] The Lie-algebraic structure of the Poisson brackets [Eq. (1)] allows the construction of Lie–Poisson operators $\{f,\circ\}$ that are associated with differentiable functions $f(\mathbf{v})$ of phase space $\mathbf{v} = (\mathbf{q}, \mathbf{p})$. The generators of the symplectic Lie algebra are the quadratic functions $q_j q_k$, $q_j p_k$, and $p_j p_k$; polynomials of higher degree generate groups of nonlinear (and generally nonglobal) transformations of phase space. When their homogeneous degree is $A > 2$, the generated transformations correspond to the $(A-1)$st-order aberrations. Symplectic groups $\mathrm{Sp}(2,\mathfrak{R})$ and $\mathrm{Sp}(4,\mathfrak{R})$ serve to classify these aberrations into irreducible multiplets that belong to the totally symmetric representation.[31] In this context, the question of whether arbitrary aberration-group elements can be realized is still wide open and probably will be answered in the negative.

The coordinate grid of phase space is the arena for the Wigner function[32] that contains both the wave field and its Fourier transform along orthogonal directions in a plane (as in Figs. 1–5); coherent states have Gaussian Wigner functions characterized by their center in phase space (a light ray of geometric optics), with squeezing (a circle becomes an ellipse under the image reduction by the DLDL arrangement of Fig. 3) and slant (Fig. 1). The harmonic evolution of a field in a waveguide corresponds to DLD arrangements in Fig. 4. Of course, nonlinear transformations, such as those that occur in optically active Kerr media, are currently of great interest; their correspondence to geometric aberrations has been explored by Atakishiyev and co-workers,[33] who used the Wigner function to characterize the wave fields deformed by geometric optical aberrations and its moments to measure their classicality.

Problems in the geometric–wave–quantum correspondence (such as operator ordering) are likely to remain: Of the same symplectic group, one has two structures (universal enveloping Lie algebras), the geometric and the wave–quantum, that are distinct. The former can be seen as the unique contraction of several models of the latter in the limit where the wavelength of light becomes zero. The linear symplectic group studied here is the domain where all of the above models are in complete correspondence.

ACKNOWLEDGMENTS

K. B. Wolf is grateful for the hospitality of the Institute of Mathematical Sciences, Chennai (formerly Madras), where this manuscript was prepared, to IIMAS–Universidad Nacional Autonoma de México (UNAM), and to project DGAPA (Dirección General de Asuntos del Personal Académico)–UNAM 104198 "Optica Matemática" for travel support. The figures were prepared by Guillermo Krötzsch of the Centro de Ciencias Físicas, UNAM–Cuernavaca.

REFERENCES

1. K. Iwasawa, "On the representations of Lie algebras," Jpn. J. Math. **19**, 513–523 (1948); "On some types of topological groups," Ann. Math. **50**, 507–558 (1948).
2. V. Bargmann, "Irreducible unitary representations of the Lorentz group," Ann. Math. **48**, 568–640 (1947); "Group representation in Hilbert spaces of analytic functions," in *Analytical Methods in Mathematical Physics*, P. Gilbert and R. G. Newton, eds. (Gordon & Breach, New York, 1970), pp. 27–63.
3. K. B. Wolf, "The symplectic groups, their parametrization and cover," in *Lie Methods in Optics*, J. Sánchez-Mondragón and K. B. Wolf, eds., Vol. 250 of Lecture Notes in Physics (Springer-Verlag, Berlin, 1986), App. A, pp. 227–238; "Representations of the algebra $\mathrm{sp}(2,\mathfrak{R})$," *ibid.*, pp. 239–247; reprinted in *Dynamical Groups and Spectrum Generating Algebras*, A. Barut, A. Bohm, and Y. Ne'eman, eds. (World Scientific, Singapore, 1989), pp. 1076–1087, 1088–1096.
4. See, e.g., H. Goldstein, *Classical Mechanics* (Addison-Wesley, Reading, Mass., 1950).
5. R. Simon and N. Mukunda, "Iwasawa decomposition for SU(1, 1) and the Gouy effect for squeezed states," Opt. Commun. **95**, 39–45 (1993).
6. R. Simon and N. Mukunda, "The two-dimensional symplectic and metaplectic groups and their universal cover," in *Symmetries in Science*, V. B. Gruber, ed. (Plenum, New York, 1993), pp. 659–689; G. S. Agarwal and R. Simon, "A simple realization of fractional Fourier transform and relation to harmonic oscillator Green's function," Opt. Commun. **110**, 23–26 (1994); K. Sundar, N. Mukunda, and R. Simon, "Coherent mode decomposition of general anisotropic Gaussian Schell model beams," J. Opt. Soc. Am. A **12**, 560–569 (1995); R. Simon and N. Mukunda, "Iwasawa decomposition in first order optics: universal treatment of shape-invariant propagation for coherent and partially coherent beams," J. Opt. Soc. Am. A **15**, 2146–2155 (1998).
7. R. Simon, N. Mukunda, and B. Dutta, "Quantum-noise matrix for multimode systems: U(n) invariance, squeezing, and normal forms," Phys. Rev. A **49**, 1567–1583 (1994).
8. K. B. Wolf, *Integral Transforms in Science and Engineering* (Plenum, New York, 1979).
9. D. F. V. James and G. S. Agarwal, "The generalized Fresnel transform and its application to optics," Opt. Commun. **126**, 207–212 (1996).
10. G. S. Agarwal and R. Simon, "An experiment for the study of the Gouy effect for squeezed states," Opt. Commun. **100**, 411–414 (1993).
11. R. Simon and N. Mukunda, "Bargmann invariant and the geometry of the Gouy effect," Phys. Rev. Lett. **70**, 880–883 (1993).
12. R. Simon, E. C. G. Sudarshan, and N. Mukunda, "Generalized rays in first order optics: transformation properties of Gaussian Schell model fields," Phys. Rev. A **29**, 3273–3279 (1984).
13. B. Dutta, N. Mukunda, R. Simon, and A. Subramaniam, "Squeezed states, photon number distribution, and U(1) invariance," J. Opt. Soc. Am. B **10**, 253–264 (1993); R. Simon, E. C. G. Sudarshan, and N. Mukunda, "Partially coherent beams and a generalized *abcd*-law," Opt. Commun. **65**, 322–328 (1988).
14. R. Simon, E. C. G. Sudarshan, and N. Mukunda, "Anisotropic Gaussian Schell model beams: passage through optical systems and associated invariants," Phys. Rev. A **31**, 2419–2434 (1985); Arvind, B. Dutta, N. Mukunda, and R. Simon, "Two-mode quantum systems: invariant classification of squeezing transformations and squeezed states," Phys. Rev. A **52**, 1609–1620 (1995).
15. E. C. G. Sudarshan, N. Mukunda, and R. Simon, "Realisation of first order optical systems using thin lenses," Opt. Acta **32**, 855–872 (1985).
16. A. Sahin, H. M. Ozaktas, and D. Mendlovic, "Optical implementations of two-dimensional fractional Fourier transforms and linear canonical transforms with arbitrary parameters," Appl. Opt. **37**, 2130–2141 (1998).
17. D. Mendlovic and H. M. Ozaktas, "Fractional Fourier transforms and their implementations. I," J. Opt. Soc. Am. A **10**, 1875–1881 (1993); H. M. Ozaktas and D. Mendlovic, "Fractional Fourier transforms and their implementations. II," J. Opt. Soc. Am. A **10**, 2522–2531 (1993); "Fractional

Fourier transform of fractional order and their optical interpretation," Opt. Commun. **101**, 163–169 (1993); "Fractional Fourier optics," J. Opt. Soc. Am. A **12**, 743–750 (1995); D. Mendlovic, Y. Bitran, R. G. Dorsch, and A. W. Lohmann, "Optical fractional correlation: experimental results," J. Opt. Soc. Am. A **12**, 1665–1670 (1995).
18. J. Shamir and N. Cohen, "Root and power transformations in optics," J. Opt. Soc. Am. A **12**, 2415–2423 (1995).
19. R. Simon, N. Mukunda, and E. C. G. Sudarshan, "Hamilton's theory of turns generalized to Sp(2, R)," Phys. Rev. Lett. **62**, 1331–1334 (1989); "The theory of screws—a new geometric representation for the group SU(1, 1)," J. Math. Phys. **30**, 1000–1006 (1989).
20. W. Hamilton, *Lectures on Quaternions* (Dublin, 1853). An excellent review of Hamilton's theory of turns for SU(2) can be found in L. C. Biedenharn and J. D. Louck, *Angular Momentum in Quantum Mechanics*, Vol. 8 of Encyclopedia of Mathematics and Its Applications (Addison-Wesley, Reading, Mass., 1881).
21. R. Gilmore, *Lie Groups, Lie Algebras, and Some of Their Applications* (Wiley, New York, 1974).
22. A. Lohmann, "The Wigner function and its optical production," Opt. Commun. **42**, 32–37 (1980); H. O. Bartelt, K.-H. Brenner, and H. Lohmann, "The Wigner distribution function and its optical production," Opt. Commun. **32**, 32–38 (1980); H. Bartelt and K.-H. Brenner, "The Wigner distribution function: an alternate signal representation in optics," Isr. J. Technol. **18**, 260–262 (1980); K.-H. Brenner and H. Lohmann, "Wigner distribution function display of complex 1D signals," Opt. Commun. **42**, 310–314 (1982).
23. R. Simon and N. Mukunda, "Twisted Gaussian Schell model beams," J. Opt. Soc. Am. A **10**, 95–109 (1993); "Twist phase in Gaussian beam optics," J. Opt. Soc. Am. A **15**, 2373–2382 (1998).
24. A. T. Friberg, E. Tervonen, and J. Turunen, "Interpretation and experimental demonstration of twisted Gaussian Schell-model beams," J. Opt. Soc. Am. A **11**, 1818–1826 (1994).
25. R. Simon, N. Mukunda, and E. C. G. Sudarshan, "Hamilton's theory of turns and a new geometrical representation for polarization optics," Pramana J. Phys. **32**, 769–792 (1989).
26. R. Simon and N. Mukunda, "Minimal three component SU(2) gadget for polarization optics," Phys. Lett. A **143**, 165–169 (1990); V. Bagini, R. Borghi, F. Gori, M. Santarsiero, F. Frezza, G. Schettini, and G. S. Spagnolo, "The Simon–Mukunda polarization gadget," Eur. J. Phys. **17**, 279–284 (1996).
27. H. Weyl, *The Theory of Groups and Quantum Mechanics*, 2nd ed. (Dover, New York, 1930).
28. M. Moshinsky and C. Quesne, "Oscillator systems," in *Proceedings of the 15th Solvay Conference in Physics (1970)* (Gordon & Breach, New York, 1974); M. Moshinsky and C. Quesne, "Linear canonical transformations and their unitary representation," J. Math. Phys. **12**, 1772–1780 (1971); C. Quesne and M. Moshinsky, "Canonical transformations and matrix elements," J. Math. Phys. **12**, 1780–1783 (1971).
29. O. Castaños, E. López-Moreno, and K. B. Wolf, "Canonical transforms for paraxial wave optics," in *Lie Methods in Optics*, J. Sánchez-Mondragón and K. B. Wolf, eds., Vol. 250 of Lecture Notes in Physics (Springer-Verlag, Berlin, 1986), pp. 159–182.
30. A. J. Dragt, E. Forest, and K. B. Wolf, "Foundations of a Lie algebraic theory of geometrical optics," in *Lie Methods in Optics*, J. Sánchez-Mondragón and K. B. Wolf, eds., Vol. 250 of Lecture Notes in Physics (Springer-Verlag, Berlin, 1986), Chap. 4, pp. 105–158.
31. K. B. Wolf, "The group-theoretical treatment of aberrating systems. III. The classification of asymmetric aberrations," J. Math. Phys. **28**, 2498–2507 (1987); "Symmetry-adapted classification of aberrations," J. Opt. Soc. Am. A **5**, 1226–1232 (1988).
32. E. Wigner, "On the quantum correction for thermodynamic equilibrium," Phys. Rev. **40**, 749–759 (1932); H.-W. Lee, "Theory and application of the quantum phase-space distribution functions," Phys. Rep. **259**, 147–211 (1995).
33. N. M. Atakishiyev, S. M. Chumakov, A. L. Rivera, and K. B. Wolf, "On the phase space description of quantum nonlinear dynamics," Phys. Lett. A **215**, 128–134 (1996); A. L. Rivera, N. M. Atakishiyev, S. M. Chumakov, and K. B. Wolf, "Evolution under polynomial Hamiltonians in quantum and optical phase spaces," Phys. Rev. A **55**, 876–889 (1997).