Meixner oscillators

Natig M. Atakishiyev
Instituto de Matemáticas, Universidad Nacional Autónoma de México
Apartado postal 273-3, 62210 Cuernavaca, Morelos, Mexico

Elchin I. Jafarov, and Shakir M. Nagiyev
Institute of Physics, Azerbaijan Academy of Sciences
H. Javid Prospekt 33, Baku 370143, Azerbaijan

Kurt Bernardo Wolf
Instituto de Investigaciones en Matemáticas Aplicadas y en Sistemas
Universidad Nacional Autónoma de México
Apartado postal 48-3, 62251 Cuernavaca, Morelos, Mexico

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Meixner oscillators have a ground state and an 'energy' spectrum that is equally spaced; they are a two-parameter family of models that satisfy a Hamiltonian equation with a *difference* operator. Meixner oscillators include as limits and particular cases the Charlier, Kravchuk and Hermite (common quantum-mechanical) harmonic oscillators. By the Sommerfeld-Watson transformation they are also related with a relativistic model of the linear harmonic oscillator, built in terms of the Meixner-Pollaczek polynomials, and their continuous weight function. We construct explicitly the corresponding coherent states with the dynamical symmetry group Sp(2,$\Re$). The reproducing kernel for the wavefunctions of these models is also found.

Keywords: Quantum mechanics; harmonic oscillators; difference operators; Meixner polynomials

El oscilador Meixner tiene un estado base y un espectro de energía uniformemente espaciado; son una familia de dos parámetros de modelos que satisfacen una equación hamiltoniana con un operador *diferencia*. Los osciladores Meixner incluyen a los osciladores armónicos de Charlier, Kravchuk y Hermite (comunes de la mecánica cuántica) como casos límite y particulares. Mediante la transformación de Sommerfeld-Watson se relacionan también con un modelo relativista del oscilador armónico lineal, construido en términos de los polinomios de Meixner-Pollaczek y sus funciones continuas de peso. Construimos explícitamente los estados coherentes correspondientes al grupo de simetría dinámica Sp(2,$\Re$). Se encuentra también el kernel de reproducción para las funciones de onda de estos modelos.

Descriptores: Mecánica cuántica; osciladores armónicos; operadores en diferencias; polinomios de Meixner

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1. Introduction

An oscillator is called *harmonic* when its oscillation period is independent of its energy. In quantum theory, this statement leads to its characterization by a Hamiltonian operator whose energy spectrum is discrete, lower-bound, and equally spaced [1],

$$H\psi_n = E_n\psi_n,$$
$$E_n \sim n + \text{constant}, \quad n = 0, 1, 2, \ldots . \tag{1}$$

Within the framework of Lie theory, this further indicates that only a few choices of operators and Hilbert spaces are available if the Hamiltonian operator is incorporated into some Lie algebra of low dimension.

If we relax the strict Schrödinger quantization rule, we find a family of harmonic oscillator models characterized by Hamiltonians that are *difference* operators (rather than *differential* operators). Their spectrum is also (1), with n either unbounded, or with an upper bound N. The wavefunctions are still continuously defined on intervals either unbounded or bounded, but the governing equation will relate their values only at discrete, equidistant points of space; the Hilbert space of wavefunctions will then also have discrete measure. Thus 'space' appears to be *discrete* rather than continuous. The two-parameter family of Meixner oscillator models, to be examined here, is harmonic. Limit and special cases of Meixner oscillators will be shown to include the Charlier, Kravchuk, and the ordinary Hermite quantum harmonic oscillator models.

We consider the two-parameter Meixner oscillator to be of interest for intertwined physical and mathematical reasons. Two physical systems, whose description leads to special cases of Meixner functions, are the relativistic model of the quantum oscillator developed in the framework of the quasipotential approach of Logunov and Tavkhelidze, and Kadyshevsky [2], and the finite optical waveguide model [3]; as well as the very well-known quantum-mechanical harmonic oscillator [1], of course. The first two models suggest a certain *discreteness* of space because they are based on *difference* equations, rather than differential ones. This is not to say, however, that space is reduced to points; but rather, that the equations of motion always relate three separate points of the continuous wavefunctions, which satisfy *discrete* orthogonality relations. Meixner oscillators seem to be a very general family of models with these characteristics.

The Hermite, Charlier and Kravchuk oscillator models are reviewed in Section 2, together with their limit relations. Their common salient feature is to possess associated raising and lowering operators for the energy quantum number n in (1). In the first two, Hermite and Charlier, the Hamiltonian operator further factorizes into the product of these raising and lowering operators; the relevant Lie algebra is the Heisenberg-Weyl one, which can be extended to the two-dimensional dynamical symplectic algebra $\mathrm{sp}(2,\Re) = \mathrm{so}(2,1) = \mathrm{su}(1,1)$.

The Meixner oscillator model is introduced in Sect. 3, using well-known properties of the Meixner polynomials and its difference equation. The dynamical symmetry is also $\mathrm{sp}(2,\Re)$. It is then natural to build the coherent states of Meixner oscillators in Sect. 4. Section 5 establishes the analogue for Meixner wavefunctions of the well-known property of the Hermite functions to self-reproduce under Fourier (and fractional Fourier) transforms. This property applies in the processing of signals by optical means [4]. Section 6 shows that limits and special cases of the Meixner oscillator are the Hermite, Charlier, and Kravchuk oscillators. In this section we discuss also how the Meixner oscillator is related the radial part of the nonrelativistic Coulomb system in quantum mechanics and a relativistic model of the linear harmonic oscillator, built in terms of the continuous Meixner-Pollaczek polynomials.

2. Hermite, Charlier and Kravchuk oscillators

In this section we collect for the reader the basic facts on the Hermite, Charlier and Kravchuk oscillators. We introduce their Hamiltonian operator and wavefunctions, as well as raising and lowering operators for each oscillator model. Finally, we show the limit relations whereby Charlier and Kravchuk "discrete" oscillators converge both to the quantum-mechanical (Hermite) harmonic oscillator.

2.1. Hermite (quantum-mechanical) oscillator

The linear harmonic oscillator in nonrelativistic quantum mechanics is governed by the well-known Hamiltonian

$$H^{\mathrm{H}}(\xi) = \frac{\hbar\omega}{2}(\xi^2 - \partial_\xi^2) = \hbar\omega\left[a^+ a + \frac{1}{2}\right], \tag{2}$$

where $\xi = \sqrt{m\omega/\hbar}\,x$ is a dimensionless coordinate (m is the mass and ω is the angular frequency of a classical oscillator); we indicate $\partial_\xi = d/d\xi$, and the creation and annihilation operators are defined as usual:

$$a^+ = \frac{1}{\sqrt{2}}(\xi - \partial_\xi), \qquad a = \frac{1}{\sqrt{2}}(\xi + \partial_\xi),$$
$$[a, a^+] = 1. \tag{3}$$

Eigenfunctions of the Hamiltonian (2) are expressed in terms of the *Hermite* polynomials $H_n(\xi)$, $n = 0, 1, 2, \ldots$. Their explicit form is

$$\begin{aligned} H_n(\xi) &= (2\xi)^n\, {}_2F_0\Big(-\frac{1}{2}n, \frac{1}{2}(1-n); -\frac{1}{\xi^2}\Big) \\ &= n! \sum_{k=0}^{[n/2]} \frac{(-1)^k (2\xi)^{n-2k}}{k!\,(n-2k)!}, \end{aligned} \tag{4}$$

where $[n/2]$ is $\frac{1}{2}n$ or $\frac{1}{2}(n-1)$ according to whether n is even or odd. Hermite polynomials are orthogonal and of square norm c_n under integration over $\xi \in \Re$, with measure $\rho^{\mathrm{H}}(\xi)\,d\xi$, where

$$\rho^{\mathrm{H}}(\xi) = e^{-\xi^2}, \qquad c_n = \sqrt{\pi}\,2^n n!. \tag{5}$$

Therefore, the normalized wavefunctions

$$\begin{aligned} \psi_n^{\mathrm{H}}(\xi) &= \sqrt{\rho^{\mathrm{H}}(\xi)/c_n}\, H_n(\xi) \\ &= \frac{1}{\sqrt{\sqrt{\pi}2^n n!}} H_n(\xi)\, e^{-\xi^2/2}, \qquad n = 0, 1, 2, \ldots, \end{aligned} \tag{6}$$

are orthonormal and complete in the Hilbert space $\mathcal{L}^2(\Re)$, commonly used in quantum mechanics, namely

$$\int_{-\infty}^{\infty} d\xi\, \psi_n^{\mathrm{H}}(\xi)\, \psi_k^{\mathrm{H}}(\xi) = \delta_{n,k},$$
$$\sum_{n=0}^{\infty} \psi_n^{\mathrm{H}}(\xi)\, \psi_n^{\mathrm{H}}(\xi') = \delta(\xi - \xi'). \tag{7}$$

Their corresponding eigenvalues under (2) define the energy spectrum of the harmonic oscillator, and are (1) in the form

$$E_n = \hbar\omega\Big(n + \frac{1}{2}\Big). \tag{8}$$

2.2. Charlier oscillator

A difference (or discrete) analogue of the linear harmonic oscillator (2), can be built on the half-line in terms of the *Charlier* polynomials $C_n(x;\mu)$, for any fixed $\mu > 0$ and $n = 0, 1, 2, \ldots$ [5]. Charlier polynomials are defined as [6, 7]

$$C_n(x;\mu) = {}_2F_0(-n, -x; \mu^{-1}) = \sum_{k=0}^{n} \frac{(-n)_k\,(-x)_k}{k!\,\mu^k}, \tag{9}$$

where $(a)_n = \Gamma(a+n)/\Gamma(a) = a(a+1)\cdots(a+n-1)$ is the shifted factorial and $\Gamma(z)$ is the Gamma function.

The Hamiltonian for the Charlier oscillator model is a difference operator [5]

$$\begin{aligned} H^{\mathrm{C}}(\xi) = \hbar\omega\Big[2\mu + \frac{1}{2} + \frac{\xi}{h_1} - \sqrt{\mu\Big(\mu + 1 + \frac{\xi}{h_1}\Big)}\, e^{h_1\partial_\xi} \\ - \sqrt{\mu\Big(\mu + \frac{\xi}{h_1}\Big)}\, e^{-h_1\partial_\xi}\Big], \end{aligned} \tag{10}$$

where by definition

$$e^{\pm y\partial_x} f(x) = f(x \pm y) \tag{11}$$

is a shift operator by y with the step $h_1 = 1/\sqrt{2\mu}$. Eigenfunctions of (10) have the same eigenvalues (8); they are orthogonal with respect to the weight function

$$\rho^{\mathrm{C}}(x) = \frac{e^{-\mu}\,\mu^x}{\Gamma(x+1)}, \tag{12}$$

and have the form

$$\psi^{\mathrm{C}}(\xi) = (-1)^n \sqrt{\frac{\mu^n}{n!}\rho^{\mathrm{C}}\Big(\mu + \frac{\xi}{h_1}\Big)}\, C_n\Big(\mu + \frac{\xi}{h_1}; \mu\Big). \tag{13}$$

It is clear from the definition (9) that the Charlier polynomials are self-dual: $C_n(x;\mu) = C_x(n;\mu)$; therefore the Charlier functions (13) satisfy *two* discrete orthogonality relations

$$\sum_{k=0}^{\infty} \psi_m^{\mathrm{C}}(\xi_k)\,\psi_n^{\mathrm{C}}(\xi_k) = \delta_{m,n}\,,$$

$$\sum_{k=0}^{\infty} \psi_k^{\mathrm{C}}(\xi_m)\,\psi_k^{\mathrm{C}}(\xi_n) = \delta_{n,m}, \tag{14}$$

where $\xi_k = (k - \mu)h_1$. These are the discrete analogues of the continuous orthogonality and the completeness relations in (7).

As in the nonrelativistic case (2), it is possible to factorize [5] the Hamiltonian (10)

$$H^{\mathrm{C}}(\xi) = \hbar\omega\Big(b^+ b + \frac{1}{2}\Big), \tag{15}$$

by means of the difference operators

$$b = \sqrt{\mu + 1 + \frac{\xi}{h_1}}\, e^{h_1\partial_\xi} - \sqrt{\mu},$$

$$b^+ = \sqrt{\mu + \frac{\xi}{h_1}}\, e^{-h_1\partial_\xi} - \sqrt{\mu}\,. \tag{16}$$

These operators satisfy the Heisenberg commutation relation

$$[b, b^+] = 1, \tag{17}$$

and their action on the wavefunctions (13) is

$$b\,\psi_n^{\mathrm{C}}(\xi) = \sqrt{n}\,\psi_{n-1}^{\mathrm{C}}(\xi)\,,$$

$$b^+\,\psi_n^{\mathrm{C}}(\xi) = \sqrt{n+1}\,\psi_{n+1}^{\mathrm{C}}(\xi)\,. \tag{18}$$

2.3. Kravchuk oscillators

Another difference analogue of the harmonic oscillator [8] can be built on the finite interval $[0, N]$, where N is some positive integer, in terms of the Kravchuk polynomials [6, 7]

$$K_n(x;p,N) = {}_2F_1(-n, -x; -N; p^{-1}) = \sum_{k=0}^{n} \frac{(-n)_k\,(-x)_k}{k!\,(-N)_k\,p^k}. \tag{19}$$

This is a family of polynomials, parametrized by $0 < p < 1$, of degree $n = 0, 1, 2, \ldots$ in the variable x.

The corresponding Kravchuk oscillator Hamiltonian is a difference operator with step $h_2 = \sqrt{2Npq}$ [8],

$$H^{\mathrm{K}}(\xi) = \hbar\omega\Big\{2p(1-p)N + \frac{1}{2} + \Big(\frac{1}{2} - p\Big)\frac{\xi}{h_2} - \sqrt{p(1-p)}\,\big[\alpha(\xi)\,e^{h_2\partial_\xi} + \alpha(\xi - h_2)\,e^{-h_2\partial_\xi}\big]\Big\}\,,$$

$$\alpha(\xi) = \sqrt{\Big(qN - \frac{\xi}{h_2}\Big)\Big(pN + 1 + \frac{\xi}{h_2}\Big)}\,. \tag{20}$$

The energy spectrum is the same as in (8), except that in the Kravchuk case there are only a finite number of energy levels $n = 0, 1, \ldots, N$.

The Kravchuk polynomials are orthogonal with respect to the binomial measure

$$\rho^{\mathrm{K}}(x) = C_N^x\, p^x\, q^{N-x}\,,$$

$$C_N^x = N!/\Gamma(x+1)\,\Gamma(N-x+1)\,. \tag{21}$$

The eigenfunctions of the difference operator (20) are

$$\psi_n^{\mathrm{K}}(\xi) = (-1)^n \sqrt{C_N^n\Big(\frac{p}{1-p}\Big)^n \rho^{\mathrm{K}}(pN + \frac{\xi}{h_2})} \times K_n(pN + \xi/h_2; p, N)\,, \tag{22}$$

where C_n^m is the binomial coefficient. The Kravchuk polynomials (19) are also self-dual, and therefore the Kravchuk functions (22) satisfy the discrete orthogonality and completeness relations over the points $\xi_j = (j - pN)h_2$:

$$\sum_{j=0}^{N} \psi_n^{\mathrm{K}}(\xi_j)\,\psi_k^{\mathrm{K}}(\xi_j) = \delta_{n,k}\,,$$

$$\sum_{j=0}^{N} \psi_j^{\mathrm{K}}(\xi_n)\,\psi_j^{\mathrm{K}}(\xi_k) = \delta_{n,k}\,, \tag{23}$$

for $n, k = 0, 1, \ldots, N$.

Now, it has been shown in [8] that the difference operators

$$A(\xi) = (1-p)\,\alpha(\xi)\,e^{h_2\partial_\xi} - pe^{-h_2\partial_\xi}\,\alpha(\xi) + \sqrt{p(1-p)}\,[(2p-1)N + 2\xi/h_2]\,, \tag{24}$$

$$A^+(\xi) = (1-p)e^{-h_2\partial_\xi}\,\alpha(\xi) - p\,\alpha(\xi)\,e^{h_2\partial_\xi} + \sqrt{p(1-p)}\,[(2p-1)N + 2\xi/h_2]\,, \tag{25}$$

together with the operator

$$A_0(\xi) = \frac{1}{\hbar\omega}\Big[H^{\mathrm{K}}(\xi) - \frac{1}{2}(N+1)\Big]\,, \tag{26}$$

close under commutation as the algebra so(3) of the rotation group,

$$[A_0, A] = -A, \qquad [A_0, A^+] = A^+, \qquad [A^+, A] = 2A_0. \tag{27}$$

The action of the operators (24–25) on the wavefunctions (22) is given by

$$A(\xi)\,\psi_n^{\mathrm{K}}(\xi) = \sqrt{n(N-n+1)}\,\psi_{n-1}^{\mathrm{K}}(\xi)\,, \tag{28}$$

$$A^+(\xi)\,\psi_n^{\mathrm{K}}(\xi) = \sqrt{(n+1)(N-n)}\,\psi_{n+1}^{\mathrm{K}}(\xi)\,. \tag{29}$$

We note that the Kravchuk oscillator was applied recently in finite (multimodal, shallow) waveguide optics [3].

2.4. Limiting cases

Among the previous models, the Kravchuk oscillator is the most general; it limits to the Charlier oscillator; in turn, the latter limits to the common Hermite harmonic oscillator [8]:
Kravchuk ⟶ Charlier. Because of the limit relation [7]

$$\lim_{N\to\infty} K_n(x;\mu/N,N) = C_n(x;\mu) \tag{30}$$

between the Kravchuk (19) and Charlier (9) polynomials, when $N\to\infty$ and $p=\mu/N\to 0$, the operators $H^{\mathrm{K}}(\xi)$, $A(\xi)/\sqrt{N}$ and $A^+(\xi)/\sqrt{N}$ reduce to the Charlier Hamiltonian (15), and the lowering and raising operators (16) for the Charlier functions, respectively. The so(3) algebra (27) in turn contracts to the Heisenberg-Weyl algebra (17).
Charlier ⟶ Hermite. In the limit when the Charlier parameter μ tends to infinity, we have [7]

$$\lim_{\mu\to\infty} h_1^{-n}\,C_n(\mu+\xi/h_1;\mu) = (-1)^n\,H_n(\xi)\,. \tag{31}$$

Similarly, in the limit $\mu\to\infty$, the operators (16) become $b\to a$, $b^+\to a^+$, and $H^{\mathrm{C}}(\xi)\to H^{\mathrm{H}}(\xi)$. The Charlier functions (12) coincide then with the Hermite functions (6), *i.e.*

$$\lim_{\mu\to\infty} h_1^{-1/2}\,\psi_n^{\mathrm{C}}(\xi) = \psi_n^{\mathrm{H}}(\xi)\,. \tag{32}$$

Kravchuk ⟶ Hermite. From the limit relations [7, 8]

$$\lim_{N\to\infty} (-1)^n\sqrt{c_N^n(p/q)^n}K_n(pN+\xi/h_2;p,N) = \frac{1}{\sqrt{2^n n!}}\,H_n(\xi)\,, \tag{33}$$

and

$$\lim_{N\to\infty} h_2^{-1}\,\rho^{\mathrm{K}}(pN+\xi/h_2) = \frac{1}{\sqrt{\pi}}\,e^{-\xi^2}\,, \tag{34}$$

it follows that

$$\lim_{N\to\infty} h_2^{-1/2}\,\psi_n^{\mathrm{K}}(\xi) = \psi_n(\xi)\,. \tag{35}$$

Also, when $N\to\infty$, the operators $H^{\mathrm{K}}(\xi)$, $A(\xi)/\sqrt{N}$ and $A^+(\xi)/\sqrt{N}$ reduce to the Hermite Hamiltonian (2), annihilation $a(\xi)$, and creation $a^+(\xi)$ operators (3) for the ordinary quantum harmonic oscillator, respectively. The so(3) algebra (27) of this finite oscillator contracts to the Heisenberg-Weyl algebra of quantum mechanics.

3. Meixner oscillators

We now organize the properties of the Meixner polynomials [6, 7] according to the scheme followed in the previous section. Known orthogonality relations for the Meixner polynomials lead to orthonormal functions and a difference Hamiltonian operator, whose spectrum is the set of energy levels (1).

3.1. Meixner polynomials and functions

The Meixner polynomials [6, 7] are Gauss hypergeometric polynomials

$$M_n(\xi;\beta,\gamma) = {}_2F_1(-n,-\xi;\beta;1-1/\gamma) = M_\xi(n;\beta,\gamma)\,. \tag{36}$$

They form a two-parameter family of polynomials, for $\beta>0$ and $0<\gamma<1$, of degree $n=0,1,2,\ldots$. Their orthogonality relation is

$$\sum_{m=0}^{\infty} \rho^{\mathrm{M}}(m)M_n(m;\beta,\gamma)M_k(m;\beta,\gamma) = d_n\delta_{nk} \tag{37}$$

with respect to the weight function and square norm

$$\rho^{\mathrm{M}}(\xi) = \frac{(\beta)_\xi\,\gamma^\xi}{\xi!}, \qquad d_n = \frac{n!}{\gamma^n(\beta)_n(1-\gamma)^\beta}. \tag{38}$$

Hence, the wavefunctions of the form

$$\psi_n^{\mathrm{M}}(\xi;\beta,\gamma) = (-1)^n\sqrt{\rho^{\mathrm{M}}(\xi)/d_n}\,M_n(\xi;\beta,\gamma), \tag{39}$$

satisfy the discrete orthogonality relations

$$\sum_{m=0}^{\infty}\psi_n^{\mathrm{M}}(m;\beta,\gamma)\,\psi_k^{\mathrm{M}}(m;\beta,\gamma) = \delta_{n,k}\,, \qquad \sum_{n=0}^{\infty}\psi_n^{\mathrm{M}}(m;\beta,\gamma)\,\psi_n^{\mathrm{M}}(m';\beta,\gamma) = \delta_{m,m'}\,, \tag{40}$$

as a consequence of (37) and the self-duality of Meixner polynomials (36). Henceforth we shall supress for brevity the super-index M from all operators and functions of the Meixner oscillator model.

The Meixner polynomials (36) satisfy the three-term recurrence relation [7]

$$[n+(n+\beta)\gamma-(1-\gamma)\xi]M_n(\xi;\beta,\gamma) = (n+\beta)\gamma M_{n+1}(\xi;\beta,\gamma)+nM_{n-1}(\xi;\beta,\gamma), \quad (41)$$

and the difference equation in the real argument

$$[\gamma(\xi+\beta)e^{\partial_\xi}+\xi e^{-\partial_\xi}-(1+\gamma)(\xi+\beta/\gamma) + (1+\gamma)(n+\beta/2)]\,M_n(\xi;\beta,\gamma)=0\,. \quad (42)$$

Hence, the functions (39) are eigenfunctions of the difference Meixner Hamiltonian operator

$$H(\xi)=\frac{1+\gamma}{1-\gamma}\,(\xi+\beta/2) - \frac{\sqrt{\gamma}}{1-\gamma}\left[\mu(\xi)e^{\partial_\xi}+\mu(\xi-1)e^{-\partial_\xi}\right], \quad (43)$$

$$\mu(\xi)=\sqrt{(\xi+1)(\xi+\beta)}\,, \quad (44)$$

with eigenvalues

$$E_n=n+\beta/2\,, \qquad n=0,1,2,\ldots\,. \quad (45)$$

3.2. Dynamical symmetry algebra Sp(2, ℜ)

As in all previous cases, we can construct the dynamical symmetry algebra (see, for example, Refs. 9 and 10) by factorizing [11, 12] the difference Hamiltonian (43). Indeed, one can verify that

$$H(\xi)=B\,B^+ +\beta/2-1, \quad (46)$$

where $B=B(\xi)$ and $B^+=B^+(\xi)$ are the difference operators

$$B=\frac{1}{\sqrt{1-\gamma}}\left[\sqrt{\xi+1}e^{\frac12\partial_\xi}-\sqrt{\gamma(\xi+\beta-1)}e^{-\frac12\partial_\xi}\right], \quad (47)$$

$$B^+=\frac{1}{\sqrt{1-\gamma}}\left[e^{-\frac12\partial_\xi}\sqrt{\xi+1}-e^{\frac12\partial_\xi}\sqrt{\gamma(\xi+\beta-1)}\right]. \quad (48)$$

It is essential to note that the factorization of the Hamiltonian (43), in contrast to the case of the harmonic oscillator (2) and the difference model (15), does not lead immediately to a closed algebra consisting of $H(\xi)$, B and B^+. To obtain such an algebra, we compute the explicit form of the commutator between the last two,

$$[B,B^+]=H(\xi)-\frac{1+\gamma}{1-\gamma}\left[\xi+\frac12(\beta-1)\right] + \frac{\sqrt{\gamma}}{1-\gamma}\left[\sqrt{\left(\xi+\beta-\frac12\right)\left(\xi+\frac32\right)}\;e^{\partial_\xi} + \sqrt{\left(\xi+\beta-\frac32\right)\left(\xi+\frac12\right)}\;e^{-\partial_\xi}\right]. \quad (49)$$

The right-hand side of (49) suggests that it is necessary to introduce new operators

$$C=\sqrt{\frac{1-\gamma}{\gamma}}B\,e^{-\frac12\partial_\xi}\sqrt{\xi+1}=\frac{\xi+1}{\sqrt{\gamma}}-e^{-\partial_\xi}\mu(\xi), \quad (50)$$

$$C^+=\sqrt{\frac{1-\gamma}{\gamma}}\sqrt{\xi+1}\,e^{\frac12\partial_\xi}B^+=\frac{\xi+1}{\sqrt{\gamma}}-\mu(\xi)e^{\partial_\xi}. \quad (51)$$

These new operators have the following commutation relations with the Hamiltonian operator

$$[H,C]=-C+\frac{1}{\sqrt{\gamma}}\,(H+1-\beta/2)\,, \quad (52)$$

$$[H,C^+]=C^+-\frac{1}{\sqrt{\gamma}}\,(H+1-\beta/2)\,. \quad (53)$$

We now build the difference operators

$$K_+=C^+-\frac{1}{\sqrt{\gamma}}(H+1-\beta/2) = \frac{\gamma}{1-\gamma}\mu(\xi)e^{\partial_\xi}+\frac{1}{1-\gamma}e^{-\partial_\xi}\mu(\xi)-\frac{2\sqrt{\gamma}}{1-\gamma}(\xi+\beta/2), \quad (54)$$

$$K_-=C-\frac{1}{\sqrt{\gamma}}\,(H+1-\beta/2) = \frac{1}{1-\gamma}\mu(\xi)e^{\partial_\xi}+\frac{\gamma}{1-\gamma}e^{-\partial_\xi}\mu(\xi)-\frac{2\sqrt{\gamma}}{1-\gamma}(\xi+\beta/2). \quad (55)$$

Together with $K_0=H$, they now form the closed Lie algebra sp(2,ℜ),

$$[K_0,K_\pm]=\pm K_\pm\,, \qquad [K_-,K_+]=2K_0\,. \quad (56)$$

The raising and lowering operators K_+ and K_- are connected by with the cartesian generators

$$K_1=-\frac{i}{2}(K_+-K_-)=\frac{i}{2}\left[\mu(\xi)e^{\partial_\xi}-e^{-\partial_\xi}\mu(\xi)\right], \quad (57)$$

$$K_2=-\frac12(K_++K_-)=-\frac{1+\gamma}{2(1-\gamma)}\left[\mu(\xi)e^{\partial_\xi}+e^{-\partial_\xi}\mu(\xi)\right] + \frac{2\sqrt{\gamma}}{1-\gamma}(\xi+\beta/2)\,. \quad (58)$$

The invariant Casimir operator in this case is

$$K^2=K_0^2-K_1^2-K_2^2 = K_0^2-K_0-K_+K_- = \frac{\beta}{2}\left(\frac{\beta}{2}-1\right)I. \quad (59)$$

The eigenvalue $-\beta/2$ of the Casimir operator K^2 determines that the model realizes the unitary irreducible representation $D^+(-\beta/2)$ of the Sp(2,ℜ) group. The eigenvalues of the compact generator $K_0(\xi)$ in such representations are bounded from below and equal to $\beta/2+n$, $n=0,1,2,\ldots$. In other words, a purely algebraic approach enables one to find the correct spectrum of the Hamiltonian $H(\xi)=K_0(\xi)$ in (45).

The action of the raising and lowering difference operators K_+ and K_- on the wavefunctions (39) is given by

$$K_+\,\psi_n(\xi;\beta,\gamma)=\kappa_{n+1}\,\psi_{n+1}(\xi;\beta,\gamma)\,,$$
$$K_-\,\psi_n(\xi;\beta,\gamma)=\kappa_n\,\psi_{n-1}(\xi;\beta,\gamma)\,, \tag{60}$$

where $\kappa_n=\sqrt{n(n+\beta-1)}$. Hence the functions $\psi_n(\xi;\beta,\gamma)$ can be obtained by n-fold application of the operator K_+ to the ground state wavefunction, that is

$$\psi_n(\xi;\beta,\gamma)=\frac{1}{\sqrt{n!(\beta)_n}}\,K_+^n\,\psi_0(\xi;\beta,\gamma)\,, \tag{61}$$

$$\psi_0(\xi;\beta,\gamma)=\sqrt{(1-\gamma)^\beta\,\rho(\xi)}\,. \tag{62}$$

3.3. Unitary equivalence in the second parameter

Observe that the eigenvalues of the Casimir operator (59), as well as the matrix elements (60) of the operators K_+ and K_-, do not depend on the second parameter, γ, of the Meixner wavefunctions. Therefore the basis functions (39), corresponding to two distinct values of the parameter γ, must be intertwined by a unitary transformation. To find its explicit form we may compare two sets of the generators K_0, K_1 and K_2 [see formulas (43) and (57,58)], corresponding to different values γ and γ'.

Introducing angles θ and θ' such that $\gamma=\tanh^2(\theta/2)$, $\gamma'=\tanh^2(\theta'/2)$ and $\delta=\theta'-\theta$, the relation between the two sets of generators is written as

$$K_0=\cosh\delta\,K_0'+\sinh\delta\,K_2'\,,$$
$$K_1=K_1'\,, \tag{63}$$
$$K_2=\sinh\delta\,K_0'+\cosh\delta\,K_2'\,.$$

This shows they are related by a boost in the 0–2 plane by the hyperbolic angle $\delta\in\Re$. Consequently, the wavefunctions (39) with different values of the parameter γ are connected by

$$\psi_n(\xi;\beta,\gamma)=e^{iK_1'\delta}\,\psi_n(\xi;\beta,\gamma')$$
$$=\sum_{k=0}^{\infty}M_{n,k}^{\gamma,\gamma'}\,\psi_k(\xi;\beta,\gamma')\,. \tag{64}$$

The last expression is the matrix form, with elements

$$M_{n,k}^{\gamma,\gamma'}=\sum_{\xi=0}^{\infty}\psi_n(\xi;\beta,\gamma)\,\psi_k(\xi;\beta,\gamma')$$
$$=(-1)^k\sqrt{\frac{(\beta)_n(\beta)_k}{n!\,k!}}\left(\tanh\frac{\delta}{2}\right)^{n+k}$$
$$\times\left(\cosh\frac{\delta}{2}\right)^{-\beta}M_n\Big(k,\beta,\tanh^2\frac{\delta}{2}\Big)\,. \tag{65}$$

In deriving (65) we have used the addition formula for the Meixner polynomials (36) given in Ref. 13, Eq. (A.6).

4. Coherent states

The dynamical symmetry of the Meixner oscillator model (43), allows us to construct two kinds of coherent states [14, 15]. Recall that in the case of harmonic oscillator (2) coherent states are defined as eigenstates of the annihilation operator $a(\xi)$ [16]. Coherent states for the model (43) can be defined either as eigenstates [14] of the lowering operator $K_-(\xi)$, or by acting on the ground state (62) with the operator $\exp[\zeta K_+(\xi)]$ [15]. This gives rise to two distinct coherent states.

4.1. The Barut-Girardello coherent states

Characterizing the Barut-Girardello coherent states by the complex number $z\in\mathcal{C}$, which is the eigenvalue under the lowering operator,

$$K_-\,\phi_z(\xi;\beta,\gamma)=z\,\phi_z(\xi;\beta,\gamma)\,, \tag{66}$$

these coherent states can be expanded in terms of the wavefunctions (39),

$$\phi_z(\xi;\beta,\gamma)=\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!(\beta)_n}}\,\psi_n(\xi;\beta,\gamma)\,. \tag{67}$$

Using the generating function [7] for the Meixner polynomials (36)

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}\,M_n(\xi;\beta,\gamma)=e^t\,{}_1F_1\Big(-\xi;\beta;\frac{1-\gamma}{\gamma}t\Big), \tag{68}$$

their explicit form is found to be

$$\phi_z(\xi;\beta,\gamma)=e^{-z\sqrt{\gamma}}\,{}_1F_1\left(-\xi;\beta;\frac{\gamma-1}{\sqrt{\gamma}}z\right)\psi_0(\xi;\beta,\gamma). \tag{69}$$

These coherent states are overcomplete and therefore nonorthogonal,

$$\sum_{\xi=0}^{\infty}\phi_z^*(\xi;\beta,\gamma)\,\phi_{z'}(\xi;\beta,\gamma)=$$
$$(z^*z')^{(1-\beta)/2}\,\Gamma(\beta)I_{\beta-1}(\sqrt{z^*z'})\,, \tag{70}$$

where $I_\nu(z)$ is the modified Bessel function.

4.2. Perelomov coherent states

The second definition of generalized coherent states is due to Perelomov [15]; it is built through the action of the group operator $\exp(\zeta K_+)$ on the ground state $\psi_0(\xi;\beta,\gamma)$:

$$\begin{aligned}\chi_\zeta(\xi;\beta,\gamma) &= (1-|\zeta|^2)^{\beta/2}\exp(\zeta K_+)\,\psi_0(\xi;\beta,\gamma)\\ &= (1-|\zeta|^2)^{\beta/2}\sum_{n=0}^{\infty}\sqrt{\frac{(\beta)_n}{n!}}\,\zeta^n\,\psi_n(\xi;\beta,\gamma)\,, \end{aligned} \tag{71}$$

where ζ is a complex number such that $|\zeta|<1$.

Using the generating function for the Meixner polynomials [6, 7],

$$\sum_{n=0}^{\infty}\frac{(\beta)_n}{n!}t^n M_n(\xi;\beta,\gamma) = \left(1-\frac{t}{\gamma}\right)^{\xi}(1-t)^{-\xi-\beta}, \tag{72}$$

we find

$$\begin{aligned}\chi_\zeta(\xi;\beta,\gamma) &= (1-|\zeta|^2)^{\beta/2}\left(1+\frac{\zeta}{\sqrt{\gamma}}\right)^{\xi}\\ &\quad\times(1+\sqrt{\gamma}\,\zeta)^{-\xi-\beta}\,\psi_0(\xi;\beta,\gamma)\,. \end{aligned} \tag{73}$$

These coherent states satisfy the relation [*cf.* (70)]

$$\begin{aligned}&\sum_{\xi=0}^{\infty}\chi^*_\zeta(\xi;\beta,\gamma)\,\chi_{\zeta'}(\xi;\beta,\gamma) = \\ &\qquad [(1-|\zeta'|^2)\,(1-|\zeta|^2)]^{\beta/2}(1-\zeta^*\zeta')^{-\beta}\,. \end{aligned} \tag{74}$$

5. Reproducing transforms

Consider the task to find a reproducing kernel for the Meixner functions (39), defined by the relation [17]

$$\sum_{\xi'=0}^{\infty}\mathcal{K}_t(\xi,\xi')\,\psi_n(\xi',\beta,\gamma) = t^n\,\psi_n(\xi;\beta,\gamma)\,. \tag{75}$$

The quantum mechanical analogue of this expression is the property of Hermite functions to reproduce under fractional Fourier transforms of angle τ for $t=e^{i\tau}$; the common Fourier transform of kernel $\exp(i\xi\xi')$ corresponds to $\tau=\pi/2$ [18]. The Fourier-Kravchuk transform has the same property on the Kravchuk functions, and has been shown recently to apply to shallow multimodal waveguides with a finite number of sensors [3].

Using the dual orthogonality relation of the Meixner functions (40), the explicit form of the kernel $\mathcal{K}_t(\xi,\xi')$ is found for $|t|<1$,

$$\mathcal{K}_t(\xi,\xi') = \sum_{n=0}^{\infty} t^n\,\psi_n(\xi;\beta,\gamma)\,\psi_n(\xi';\beta,\gamma)\,. \tag{76}$$

It is a bilinear generating function for the Meixner functions. By the definition (76), the reproducing kernel $\mathcal{K}_t(\xi,\xi')$ is symmetric with respect to exchange of ξ and ξ', and because of the orthogonality relation (40) it has the property

$$\sum_{\xi'=0}^{\infty}\mathcal{K}_t(\xi,\xi')\,\mathcal{K}_{t'}(\xi',\xi'') = \mathcal{K}_{tt'}(\xi,\xi'')\,. \tag{77}$$

Reproducing kernels for the Charlier (12) and Kravchuk (22) functions have been discussed in [19], whereas the cases of the q-Hermite and Askey-Wilson polynomials have been considered in [20] and [21, 22], respectively.

Substituting (39) in (76), we can write

$$\begin{aligned}\mathcal{K}_t(\xi,\xi') &= \sqrt{\rho(\xi)\,\rho(\xi')}\,(1-\gamma)^{\beta}\\ &\quad\times\sum_{n=0}^{\infty}\frac{(\beta)_n}{n!}(\gamma t)^n M_n(\xi;\beta,\gamma)\,M_n(\xi';\beta,\gamma)\,. \end{aligned} \tag{78}$$

The sum over n in (78) is the bilinear generating function (Poisson kernel) for the Meixner polynomials [23],

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{(\beta)_n}{n!}t^n M_n(\xi;\beta,\gamma)\,M_n(\xi';\beta,\gamma) = (1-t)^{-\beta-\xi-\xi'}\\ &\quad\times\left(1-\frac{t}{\gamma}\right)^{\xi+\xi'}{}_2F_1\left[-\xi,-\xi';\beta;\frac{t(1-\gamma^2)}{(t-\gamma)^2}\right]. \end{aligned} \tag{79}$$

Thus the kernel $\mathcal{K}_t(\xi,\xi')$ is written as

$$\begin{aligned}\mathcal{K}_t(\xi,\xi') &= \sqrt{\rho(\xi)\,\rho(\xi')}\,\frac{(1-\gamma)^{\beta}(1-t)^{\xi+\xi'}}{(1-\gamma t)^{\xi+\xi'+\beta}}\\ &\quad\times{}_2F_1\left[-\xi,-\xi';\beta;\frac{t(1-\gamma^2)}{\gamma(1-t)^2}\right]. \end{aligned} \tag{80}$$

For integer ξ and ξ' we have the limit

$$\lim_{t\to 1^-}\mathcal{K}_t(\xi,\xi') = \delta_{\xi,\xi'}\,. \tag{81}$$

In this limit, the relation (76) coincides with the dual orthogonality (40) of the Meixner functions.

The limit of (80) when $t\to i$ ($\tau\to\pi/2$) corresponds to a discrete analogue of the classical Fourier-Bessel transform; whereas the latter integrates over the nonnegative half-axis, the former sums over the integer points $\xi=0,1,\ldots$. The limit is

$$\begin{aligned}\mathcal{K}_i(\xi,\xi') &= \lim_{t\to i}\mathcal{K}_t(\xi,\xi')\\ &= \sqrt{\rho(\xi)\,\rho(\xi')}\,\frac{(-2i)^{(\xi+\xi')/2}(1-\gamma)^{\beta}}{(1-i\gamma)^{\xi+\xi'+\beta}}\\ &\quad\times{}_2F_1\left[-\xi,-\xi';-\beta;-\frac{(1-\gamma)^2}{2\gamma}\right]. \end{aligned} \tag{82}$$

It is easy to verify that for integer ξ and ξ'',

$$\sum_{\xi'=0}^{\infty}\mathcal{K}_i(\xi,\xi')\,\mathcal{K}_i(\xi',\xi'') = \delta_{\xi,\xi''}\,. \tag{83}$$

6. Limit and special cases

The difference model of the Meixner linear harmonic oscillator family (43) contains as limit and particular cases all the models of Hermite (2), Charlier (10) and Kravchuk (20). We make these limits explicit below. Here we discuss also the corresponding relations with the radial part of the nonrelativistic Coulomb system in quantum mechanics and a relativistic model of the linear oscillator, built in terms of the continuous Meixner-Pollaczek polynomials.

6.1. Meixner ⟶ Hermite

From the recurrence relation for the Meixner polynomials (41), it can be show that the following limit to the Hermite polynomials holds:

$$\lim_{\nu\to\infty}(2\nu)^{n/2}M_n\left(\frac{\nu+\sqrt{2\nu}\xi}{1-\gamma};\frac{\nu}{\gamma},\gamma\right)=(-1)^nH_n(\xi). \qquad (84)$$

Furthermore, measures and normalization coefficients relate as

$$\lim_{\nu\to\infty}\sqrt{2\nu}(1-\gamma)^{\nu/\gamma-1}\rho\left(\frac{\nu+\sqrt{2\nu}\xi}{1-\gamma}\right)=\frac{1}{\sqrt{\pi}}e^{-\xi^2}, \qquad (85)$$

$$\lim_{\nu\to\infty}(2\nu)^{n/2}\,(1-\gamma)^{\nu/2\gamma}d_n=\sqrt{2^n n!}\,. \qquad (86)$$

The wavefunctions (39) with argument $(\nu+\sqrt{2\nu}\xi)/(1-\gamma)$ and $\beta=\nu/\gamma$, coincide in the limit $\nu\to\infty$ with the wavefunctions of the linear harmonic oscillator (6), *i.e.*,

$$\lim_{\nu\to\infty}\frac{(2\nu)^{1/4}}{(1-\gamma)^{1/2}}\psi_n\left(\frac{\nu+\sqrt{2\nu}\xi}{1-\gamma};\frac{\nu}{\gamma},\gamma\right)=\psi_n^{\rm H}(\xi)\,. \qquad (87)$$

The combination $K_0(\xi)-\nu/2\gamma$ reproduces, in the same limit, the product $a^+(\xi)a(\xi)$, whereas the matrix elements of $\sqrt{\gamma/\nu}K_\pm(\xi)$ converge to the creation and annihilation operators $a^+(\xi)$ and $a(\xi)$, respectively. The Meixner oscillator family (43) thus contains as a limit case the linear harmonic oscillator (2) of quantum mechanics.

6.2. Meixner ⟶ Charlier

It is known that the Meixner (36) and Charlier (9) polynomials are connected by the limit relation [6, 7]

$$\lim_{\beta\to\infty}M_n(\xi;\beta,\mu/\beta)=C_n(\xi,\mu)\,. \qquad (88)$$

Hence in the limit when $\beta\to\infty$ and $\gamma=\mu/\beta\to 0$, from (82) one obtains the reproducing kernel for the Charlier functions [19]:

$$\mathcal{K}^{\rm C}(\xi,\xi')=\lim_{\beta\to\infty,\ \beta\gamma=\mu}\mathcal{K}_i(\xi,\xi')$$
$$=e^{-(1-i)\mu}\sqrt{\frac{(-2i\mu)^{\xi+\xi'}}{\xi!\,\xi'!}}$$
$$\times\,{}_2F_0\left[-\xi,-\xi';-\frac{1}{2\mu}\right]. \qquad (89)$$

Using the limit relation (88) it is easy to check that

$$\lim_{\beta\to\infty}\psi_n(\mu+\xi/h_1;\beta,\mu/\beta)=\psi_n^{\rm C}(\xi)\,. \qquad (90)$$

Hence the wavefunctions (39) with argument $\mu+\xi/h_1$ and parameter $\gamma=\mu/\beta$ coincide, in the limit when $\beta\to\infty$, with the wave funcitons of the difference (discrete) model of the Charlier oscillator (10). In the same limit, the combination $H(s)+\frac{1}{2}(1-\beta)$ reproduces $H^{\rm C}(\xi)/\hbar\omega$, whereas $\beta^{-1/2}K_\pm(\xi)$ tend to the raising and lowering operators B^+ and B, respectively.

6.3. Meixner ⟶ Kravchuk

The Kravchuk polynomials (19) are also a particular case of the Meixner polynomials (36), with the parameters $\beta=-N$ and $\gamma=-p/(1-p)$, that is,

$$K_n(\xi;p,N)=M_n(\xi;-N,p/(p-1))\,. \qquad (91)$$

In this case, from (82) one obtains the reproducing kernel for the Kravchuk functions (22) [19],

$$\mathcal{K}^{\rm K}(\xi,\xi')=\sqrt{(-2ipq)^{\xi+\xi'}\,C_N^{\xi}C_N^{\xi'}}\,\Big[1-(1-i)p\Big]^{N-\xi-\xi'}$$
$$\times\,{}_2F_1\left[-\xi,-\xi';-n;\frac{1}{2p(1-p)}\right]. \qquad (92)$$

As follows from the relation (91) for $\beta=-N$, $\gamma=-p/(1-p)$ and argument $pN+\sqrt{2p(1-p)N}\xi$, the model (43) coincides with the difference model of the Kravchuk oscillator (20).

6.4. Meixner ⟶ Laguerre

The limit relation [7, 24]

$$\lim_{h\to 0}M_n(x/h;\beta,1-h)=\frac{n!}{(\beta)_n}L_n^{\beta-1}(x), \qquad (93)$$

where $L_n^\alpha(x)$ are the Laguerre polynomials, enables us to consider the nonrelativistic Coulomb system as another limit case of the difference model (43). Indeed, from (39) and (93), it follows that

$$\lim_{h\to 0}\frac{1}{\sqrt{h}}\psi_n(2kr/h;2l+2,1-h)=\sqrt{2kr}R_{n,l}(2kr), \qquad (94)$$

where

$$R_{n,l}(x)=(-1)^n\sqrt{\frac{n!}{(n+2l+1)!}}x^le^{-x/2}L_n^{2l+1}(x) \qquad (95)$$

is the radial wavefunction of the Coulomb system (see, for example, Ref. 25), r is the radial variable, l and $n+l+1$ are orbital and principal quantum numbers respectively, and $k=me^2/\hbar^2(n+l+1)$.

In the limit $h \to 0$, the generators $K_0(x)$ and $K_\pm(x)$, where $x = 2kr/h$, reproduce the well-known generators of the dynamical symmetry algebra su(1,1) for the nonrelativistic Coulomb model [9]

$$J_0 = -\frac{1}{2k}\left[r\partial_r^2 + \partial_r - \frac{(l+1/2)^2}{r} - k^2 r\right], \qquad (96)$$

$$J_\pm = -J_0 + kr \mp (r\partial_r + 1/2). \qquad (97)$$

This connection between the Meixner polynomials and radial wavefunctions for the nonrelativistic Coulomb system can be used for constructing a q-analogue of the Coulomb wavefunctions in terms of the q-Meixner polynomials (see Refs. 26 and 27).

6.5. Meixner-Pollaczek (relativistic) oscillators

There is the family of Meixner-Pollaczek polynomials

$$P_n^\lambda(x;\phi) = \frac{(2\lambda)_n}{n!}\, e^{-in\phi} F(-n, \lambda - ix; 2\lambda; 1 - e^{2i\phi}), \qquad (98)$$

which satisfy the orthogonality relation

$$\int_{-\infty}^{\infty} P_n^\lambda(\xi;\phi)\, P_{n'}^\lambda(\xi;\phi)\, \rho^{\mathrm{P}}(\xi)\, d\xi = \delta_{n,n'}\, \frac{\Gamma(2\lambda+n)}{n!} \qquad (99)$$

with respect to a continuous measure with the weight

$$\rho^{\mathrm{P}}(\xi) = \frac{1}{2\pi}(2\sin\phi)^{2\lambda}|\Gamma(\lambda + i\xi)|^2 \exp[(2\phi - \pi)\xi]. \qquad (100)$$

The reason why we mention these polynomials here is the following. In Ref. 28 it was shown that the Meixner polynomials $M_n(\xi;\beta,\gamma)$ and the Meixner-Pollaczek polynomials (98) are in fact interrelated by

$$P_n^\lambda(\xi;\phi) = \frac{e^{-in\phi}}{n!}(2\lambda)_n M_n(i\xi - \lambda; 2\lambda, e^{-2i\phi}). \qquad (101)$$

The transition from the discrete orthogonality (37) for the Meixner polynomials $M_n(\xi;\beta,\gamma)$ to the continuous one (99) is analogous to the well-known Sommerfeld-Watson transformation in optics and quantum theory of scattering.

In the relativistic model of the linear harmonic oscillator, proposed in [29], the wavefunctions in configuration space are expressed in terms of the Meixner-Pollaczek polynomials (98) and their weight function (100) with the specific value of the parameter $\phi = 2/\pi$. The same model in the homogeneous external field gx corresponds to the value of the parameter ϕ given by $\arccos(g/mc\omega)$, where m and ω have the same meaning as in the classical case, and c is the velocity of light [29, 30]. In other words, the relation (101) gives the connection between the relativistic harmonic oscillator and the Meixner oscillator, discussed in this paper.

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